$$\frac{C_1B_1}{AC_1}, \frac{C_2B_2}{AC_2}, \frac{C_3B_3}{AC_3}, \ldots$$

where $C_1B_1 = s_1 - s_0$ is the distance travelled by the body in the time interval $h_1 = AC_1$, etc. From the Fig 13.1 it is safe to conclude that this latter sequence approaches the slope of the tangent to the curve at point A. In other words, the instantaneous velocity $v(t)$ of a body at time $t = 2$ is equal to the slope of the tangent of the curve $s = 4.9t^2$ at $t = 2$.

13.3 Limits

The above discussion clearly points towards the fact that we need to understand limiting process in greater clarity. We study a few illustrative examples to gain some familiarity with the concept of limits.

Consider the function $f(x) = x^2$. Observe that as x takes values very close to 0, the value of $f(x)$ also moves towards 0 (See Fig 2.10 Chapter 2). We say

$$\lim_{x \to 0} f(x) = 0$$

(to be read as limit of $f(x)$ as x tends to zero equals zero). The limit of $f(x)$ as x tends to zero is to be thought of as the value $f(x)$ should assume at $x = 0$.

In general as $x \to a$, $f(x) \to l$, then l is called *limit of the function* $f(x)$ which is symbolically written as $\lim_{x \to a} f(x) = l$.

Consider the following function $g(x) = |x|$, $x \neq 0$. Observe that $g(0)$ is not defined. Computing the value of $g(x)$ for values of x very near to 0, we see that the value of $g(x)$ moves towards 0. So, $\lim_{x \to 0} g(x) = 0$. This is intuitively clear from the graph of $y = |x|$ for $x \neq 0$. (See Fig 2.13, Chapter 2).

Consider the following function.

$$h(x) = \frac{x^2 - 4}{x - 2}, \ x \neq 2.$$

Compute the value of $h(x)$ for values of x very near to 2 (but not at 2). Convince yourself that all these values are near to 4. This is somewhat strengthened by considering the graph of the function $y = h(x)$ given here (Fig 13.2).

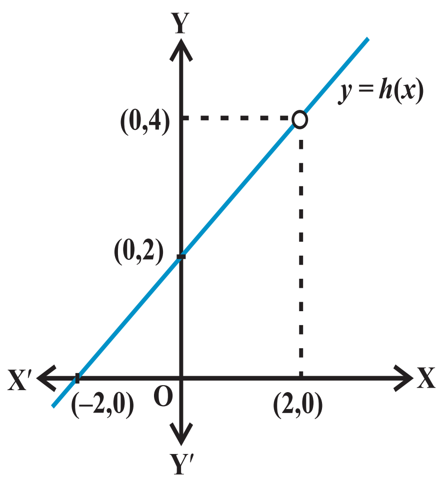

Fig 13.2

In all these illustrations the value which the function should assume at a given point $x = a$ did not really depend on how is x tending to a. Note that there are essentially two ways x could approach a number a either from left or from right, i.e., all the values of x near a could be less than a or could be greater than a. This naturally leads to two limits – the *right hand limit* and the *left hand limit*. *Right hand limit* of a function $f(x)$ is that value of $f(x)$ which is dictated by the values of $f(x)$ when x tends to a from the right. Similarly, the *left hand limit*. To illustrate this, consider the function

$$f(x) = \begin{cases} 1, & x \leq 0 \\ 2, & x > 0 \end{cases}$$

Graph of this function is shown in the Fig 13.3. It is clear that the value of f at 0 dictated by values of $f(x)$ with $x \leq 0$ equals 1, i.e., the left hand limit of $f(x)$ at 0 is

$$\lim_{x \to 0^-} f(x) = 1 .$$

Similarly, the value of f at 0 dictated by values of $f(x)$ with $x > 0$ equals 2, i.e., the right hand limit of $f(x)$ at 0 is

$$\lim_{x \to 0^+} f(x) = 2 .$$

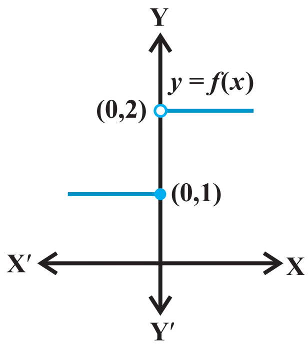

Fig 13.3

In this case the right and left hand limits are different, and hence we say that the limit of $f(x)$ as x tends to zero does not exist (even though the function is defined at 0).

Summary

We say $\lim_{x \to a^-} f(x)$ is the expected value of f at $x = a$ given the values of f near x to the left of a. This value is called the *left hand limit* of f at a.

We say $\lim_{x \to a^+} f(x)$ is the expected value of f at $x = a$ given the values of f near x to the right of a. This value is called the *right hand limit* of $f(x)$ at a.

If the right and left hand limits coincide, we call that common value as the limit of $f(x)$ at $x = a$ and denote it by $\lim_{x \to a} f(x)$.

Illustration 1 Consider the function $f(x) = x + 10$. We want to find the limit of this function at $x = 5$. Let us compute the value of the function $f(x)$ for x very near to 5. Some of the points near and to the left of 5 are 4.9, 4.95, 4.99, 4.995. . ., etc. Values of the function at these points are tabulated below. Similarly, the real number 5.001, 5.01, 5.1 are also points near and to the right of 5. Values of the function at these points are also given in the Table 13.4.

Table 13.4

x	4.9	4.95	4.99	4.995	5.001	5.01	5.1
$f(x)$	14.9	14.95	14.99	14.995	15.001	15.01	15.1

From the Table 13.4, we deduce that value of $f(x)$ at $x = 5$ should be greater than 14.995 and less than 15.001 assuming nothing dramatic happens between $x = 4.995$ and 5.001. It is reasonable to assume that the value of the $f(x)$ at $x = 5$ as dictated by the numbers to the left of 5 is 15, i.e.,

$$\lim_{x \to 5^-} f(x) = 15 .$$

Similarly, when x approaches 5 from the right, $f(x)$ should be taking value 15, i.e.,

$$\lim_{x \to 5^+} f(x) = 15 .$$

Hence, it is likely that the left hand limit of $f(x)$ and the right hand limit of $f(x)$ are both equal to 15. Thus,

$$\lim_{x \to 5^-} f(x) = \lim_{x \to 5^+} f(x) = \lim_{x \to 5} f(x) = 15 .$$

This conclusion about the limit being equal to 15 is somewhat strengthened by seeing the graph of this function which is given in Fig 2.16, Chapter 2. In this figure, we note that as x approaches 5 from either right or left, the graph of the function $f(x) = x + 10$ approaches the point (5, 15).

We observe that the value of the function at $x = 5$ also happens to be equal to 15.

Illustration 2 Consider the function $f(x) = x^3$. Let us try to find the limit of this function at $x = 1$. Proceeding as in the previous case, we tabulate the value of $f(x)$ at x near 1. This is given in the Table 13.5.

Table 13.5

x	0.9	0.99	0.999	1.001	1.01	1.1
$f(x)$	0.729	0.970299	0.997002999	1.003003001	1.030301	1.331

From this table, we deduce that value of $f(x)$ at $x = 1$ should be greater than 0.997002999 and less than 1.003003001 assuming nothing dramatic happens between

x = 0.999 and 1.001. It is reasonable to assume that the value of the $f(x)$ at x = 1 as dictated by the numbers to the left of 1 is 1, i.e.,

$$\lim_{x \to 1^-} f(x) = 1.$$

Similarly, when x approaches 1 from the right, $f(x)$ should be taking value 1, i.e.,

$$\lim_{x \to 1^+} f(x) = 1.$$

Hence, it is likely that the left hand limit of $f(x)$ and the right hand limit of $f(x)$ are both equal to 1. Thus,

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = \lim_{x \to 1} f(x) = 1.$$

This conclusion about the limit being equal to 1 is somewhat strengthened by seeing the graph of this function which is given in Fig 2.11, Chapter 2. In this figure, we note that as x approaches 1 from either right or left, the graph of the function $f(x) = x^3$ approaches the point (1, 1).

We observe, again, that the value of the function at x = 1 also happens to be equal to 1.

Illustration 3 Consider the function $f(x)$ = 3x. Let us try to find the limit of this function at x = 2. The following Table 13.6 is now self-explanatory.

Table 13.6

x	1.9	1.95	1.99	1.999	2.001	2.01	2.1
$f(x)$	5.7	5.85	5.97	5.997	6.003	6.03	6.3

As before we observe that as x approaches 2 from either left or right, the value of $f(x)$ seem to approach 6. We record this as

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) = \lim_{x \to 2} f(x) = 6$$

Its graph shown in Fig 13.4 strengthens this fact.

Here again we note that the value of the function at x = 2 coincides with the limit at x = 2.

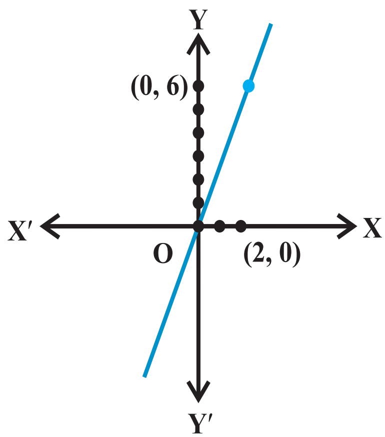

Fig 13.4

Illustration 4 Consider the constant function $f(x)$ = 3. Let us try to find its limit at x = 2. This function being the constant function takes the same value (3, in this case) everywhere, i.e., its value at points close to 2 is 3. Hence

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = \lim_{x\to 2} f(x) = 3$$

Graph of $f(x) = 3$ is anyway the line parallel to x-axis passing through (0, 3) and is shown in Fig 2.9, Chapter 2. From this also it is clear that the required limit is 3. In fact, it is easily observed that $\lim_{x\to a} f(x) = 3$ for any real number a.

Illustration 5 Consider the function $f(x) = x^2 + x$. We want to find $\lim_{x\to 1} f(x)$. We tabulate the values of $f(x)$ near $x = 1$ in Table 13.7.

Table 13.7

x	0.9	0.99	0.999	1.01	1.1	1.2
$f(x)$	1.71	1.9701	1.997001	2.0301	2.31	2.64

From this it is reasonable to deduce that

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = \lim_{x\to 1} f(x) = 2.$$

From the graph of $f(x) = x^2 + x$ shown in the Fig 13.5, it is clear that as x approaches 1, the graph approaches (1, 2).

Here, again we observe that the

$$\lim_{x\to 1} f(x) = f(1)$$

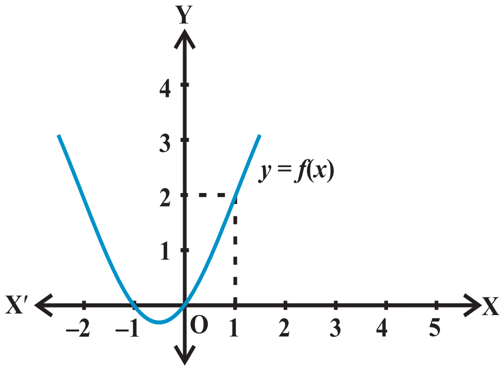

Fig 13.5

Now, convince yourself of the following three facts:

$$\lim_{x\to 1} x^2 = 1, \quad \lim_{x\to 1} x = 1 \text{ and } \lim_{x\to 1} x + 1 = 2$$

Then $$\lim_{x\to 1} x^2 + \lim_{x\to 1} x = 1 + 1 = 2 = \lim_{x\to 1}\left[x^2 + x\right].$$

Also $$\lim_{x\to 1} x.\ \lim_{x\to 1}(x+1) = 1.2 = 2 = \lim_{x\to 1}\left[x(x+1)\right] = \lim_{x\to 1}\left[x^2 + x\right].$$

Illustration 6 Consider the function $f(x) = \sin x$. We are interested in $\lim_{x \to \frac{\pi}{2}} \sin x$, where the angle is measured in radians.

Here, we tabulate the (approximate) value of $f(x)$ near $\frac{\pi}{2}$ (Table 13.8). From this, we may deduce that

$$\lim_{x \to \frac{\pi}{2}^-} f(x) = \lim_{x \to \frac{\pi}{2}^+} f(x) = \lim_{x \to \frac{\pi}{2}} f(x) = 1.$$

Further, this is supported by the graph of $f(x) = \sin x$ which is given in the Fig 3.8 (Chapter 3). In this case too, we observe that $\lim_{x \to \frac{\pi}{2}} \sin x = 1$.

Table 13.8

x	$\frac{\pi}{2} - 0.1$	$\frac{\pi}{2} - 0.01$	$\frac{\pi}{2} + 0.01$	$\frac{\pi}{2} + 0.1$
$f(x)$	0.9950	0.9999	0.9999	0.9950

Illustration 7 Consider the function $f(x) = x + \cos x$. We want to find the $\lim_{x \to 0} f(x)$.

Here we tabulate the (approximate) value of $f(x)$ near 0 (Table 13.9).

Table 13.9

x	– 0.1	– 0.01	– 0.001	0.001	0.01	0.1
$f(x)$	0.9850	0.98995	0.9989995	1.0009995	1.00995	1.0950

From the Table 13.9, we may deduce that

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = \lim_{x \to 0} f(x) = 1$$

In this case too, we observe that $\lim_{x \to 0} f(x) = f(0) = 1$.

Now, can you convince yourself that

$$\lim_{x \to 0} [x + \cos x] = \lim_{x \to 0} x + \lim_{x \to 0} \cos x \text{ is indeed true?}$$

Illustration 8 Consider the function $f(x)=\frac{1}{x^2}$ for $x>0$. We want to know $\lim_{x\to 0} f(x)$.

Here, observe that the domain of the function is given to be all positive real numbers. Hence, when we tabulate the values of $f(x)$, it does not make sense to talk of x approaching 0 from the left. Below we tabulate the values of the function for positive x close to 0 (in this table n denotes any positive integer).

From the Table 13.10 given below, we see that as x tends to 0, $f(x)$ becomes larger and larger. What we mean here is that the value of $f(x)$ may be made larger than any given number.

Table 13.10

x	1	0.1	0.01	10^{-n}
$f(x)$	1	100	10000	10^{2n}

Mathematically, we say

$$\lim_{x\to 0} f(x)=+\infty$$

We also remark that we will not come across such limits in this course.

Illustration 9 We want to find $\lim_{x\to 0} f(x)$, where

$$f(x)=\begin{cases} x-2, & x<0 \\ 0\quad , & x=0 \\ x+2, & x>0 \end{cases}$$

As usual we make a table of x near 0 with $f(x)$. Observe that for negative values of x we need to evaluate $x - 2$ and for positive values, we need to evaluate $x + 2$.

Table 13.11

x	– 0.1	– 0.01	– 0.001	0.001	0.01	0.1
$f(x)$	– 2.1	– 2.01	– 2.001	2.001	2.01	2.1

From the first three entries of the Table 13.11, we deduce that the value of the function is decreasing to –2 and hence.

$$\lim_{x\to 0^-} f(x)=-2$$

From the last three entires of the table we deduce that the value of the function is increasing from 2 and hence

$$\lim_{x \to 0^+} f(x) = 2$$

Since the left and right hand limits at 0 do not coincide, we say that the limit of the function at 0 does not exist.

Graph of this function is given in the Fig13.6. Here, we remark that the value of the function at $x = 0$ is well defined and is, indeed, equal to 0, but the limit of the function at $x = 0$ is not even defined.

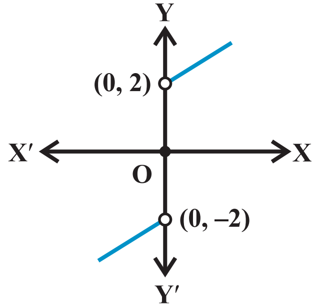

Fig 13.6

Illustration 10 As a final illustration, we find $\lim_{x \to 1} f(x)$, where

$$f(x) = \begin{cases} x+2 & x \neq 1 \\ 0 & x = 1 \end{cases}$$

Table 13.12

x	0.9	0.99	0.999	1.001	1.01	1.1
$f(x)$	2.9	2.99	2.999	3.001	3.01	3.1

As usual we tabulate the values of $f(x)$ for x near 1. From the values of $f(x)$ for x less than 1, it seems that the function should take value 3 at $x = 1$., i.e.,

$$\lim_{x \to 1^-} f(x) = 3.$$

Similarly, the value of $f(x)$ should be 3 as dictated by values of $f(x)$ at x greater than 1. i.e.

$$\lim_{x \to 1^+} f(x) = 3.$$

But then the left and right hand limits coincide and hence

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = \lim_{x \to 1} f(x) = 3.$$

Graph of function given in Fig 13.7 strengthens our deduction about the limit. Here, we

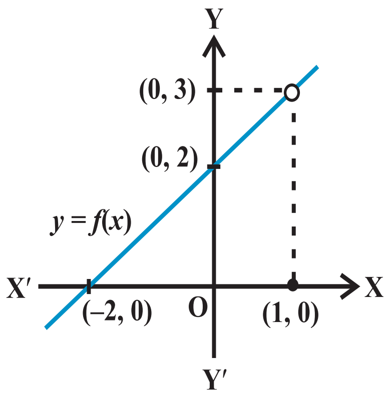

Fig 13.7

note that in general, at a given point the value of the function and its limit may be different (even when both are defined).

13.3.1 ***Algebra of limits*** In the above illustrations, we have observed that the limiting process respects addition, subtraction, multiplication and division as long as the limits and functions under consideration are well defined. This is not a coincidence. In fact, below we formalise these as a theorem without proof.

Theorem 1 Let f and g be two functions such that both $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist. Then

(i) Limit of sum of two functions is sum of the limits of the functions, i.e.,

$$\lim_{x\to a} [f(x) + g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x).$$

(ii) Limit of difference of two functions is difference of the limits of the functions, i.e.,

$$\lim_{x\to a} [f(x) - g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x).$$

(iii) Limit of product of two functions is product of the limits of the functions, i.e.,

$$\lim_{x\to a} [f(x) \cdot g(x)] = \lim_{x\to a} f(x) . \lim_{x\to a} g(x).$$

(iv) Limit of quotient of two functions is quotient of the limits of the functions (whenever the denominator is non zero), i.e.,

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$$

Note In particular as a special case of (iii), when g is the constant function such that $g(x) = \lambda$, for some real number λ, we have

$$\lim_{x\to a}\left[(\lambda.f)(x)\right] = \lambda.\lim_{x\to a} f(x).$$

In the next two subsections, we illustrate how to exploit this theorem to evaluate limits of special types of functions.

13.3.2 ***Limits of polynomials and rational functions*** A function f is said to be a polynomial function of degree n $f(x) = a_0 + a_1x + a_2x^2 + . . . + a_nx^n$, where a_is are real numbers such that $a_n \neq 0$ for some natural number n.

We know that $\lim_{x\to a} x = a$. Hence

$$\lim_{x \to a} x^2 = \lim_{x \to a}(x.x) = \lim_{x \to a} x.\lim_{x \to a} x = a.\,a = a^2$$

An easy exercise in induction on n tells us that

$$\lim_{x \to a} x^n = a^n$$

Now, let $f(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n$ be a polynomial function. Thinking of each of $a_0, a_1\,x, a_2x^2, ..., a_n\,x^n$ as a function, we have

$$\begin{aligned}\lim_{x \to a} f(x) &= \lim_{x \to a}\left[a_0 + a_1x + a_2x^2 + ... + a_nx^n\right]\\ &= \lim_{x \to a} a_0 + \lim_{x \to a} a_1x + \lim_{x \to a} a_2x^2 + ... + \lim_{x \to a} a_nx^n\\ &= a_0 + a_1\lim_{x \to a} x + a_2\lim_{x \to a} x^2 + ... + a_n\lim_{x \to a} x^n\\ &= a_0 + a_1a + a_2a^2 + ... + a_na^n\\ &= f(a)\end{aligned}$$

(Make sure that you understand the justification for each step in the above!)

A function f is said to be a rational function, if $f(x) = \dfrac{g(x)}{h(x)}$, where $g(x)$ and $h(x)$ are polynomials such that $h(x) \neq 0$. Then

$$\lim_{x \to a} f(x) = \lim_{x \to a}\frac{g(x)}{h(x)} = \frac{\lim_{x \to a} g(x)}{\lim_{x \to a} h(x)} = \frac{g(a)}{h(a)}$$

However, if $h(a) = 0$, there are two scenarios – (i) when $g(a) \neq 0$ and (ii) when $g(a) = 0$. In the former case we say that the limit does not exist. In the latter case we can write $g(x) = (x - a)^k g_1(x)$, where k is the maximum of powers of $(x - a)$ in $g(x)$ Similarly, $h(x) = (x - a)^l h_1(x)$ as $h(a) = 0$. Now, if $k > l$, we have

$$\lim_{x \to a} f(x) = \frac{\lim_{x \to a} g(x)}{\lim_{x \to a} h(x)} = \frac{\lim_{x \to a}(x - a)^k g_1(x)}{\lim_{x \to a}(x - a)^l h_1(x)}$$

$$= \frac{\lim\limits_{x \to a} (x-a)^{(k-l)} g_1(x)}{\lim\limits_{x \to a} h_1(x)} = \frac{0.g_1(a)}{h_1(a)} = 0$$

If $k < l$, the limit is not defined.

Example 1 Find the limits: (i) $\lim\limits_{x \to 1} \left[x^3 - x^2 + 1\right]$ (ii) $\lim\limits_{x \to 3} \left[x(x+1)\right]$

(iii) $\lim\limits_{x \to -1} \left[1 + x + x^2 + ... + x^{10}\right]$.

Solution The required limits are all limits of some polynomial functions. Hence the limits are the values of the function at the prescribed points. We have

(i) $\lim\limits_{x \to 1} [x^3 - x^2 + 1] = 1^3 - 1^2 + 1 = 1$

(ii) $\lim\limits_{x \to 3} \left[x(x+1)\right] = 3(3+1) = 3(4) = 12$

(iii) $\lim\limits_{x \to -1} \left[1 + x + x^2 + ... + x^{10}\right] = 1 + (-1) + (-1)^2 + ... + (-1)^{10}$

$$= 1 - 1 + 1 ... + 1 = 1.$$

Example 2 Find the limits:

(i) $\lim\limits_{x \to 1} \left[\dfrac{x^2 + 1}{x + 100}\right]$ (ii) $\lim\limits_{x \to 2} \left[\dfrac{x^3 - 4x^2 + 4x}{x^2 - 4}\right]$

(iii) $\lim\limits_{x \to 2} \left[\dfrac{x^2 - 4}{x^3 - 4x^2 + 4x}\right]$ (iv) $\lim\limits_{x \to 2} \left[\dfrac{x^3 - 2x^2}{x^2 - 5x + 6}\right]$

(v) $\lim\limits_{x \to 1} \left[\dfrac{x-2}{x^2 - x} - \dfrac{1}{x^3 - 3x^2 + 2x}\right]$.

Solution All the functions under consideration are rational functions. Hence, we first evaluate these functions at the prescribed points. If this is of the form $\frac{0}{0}$, we try to rewrite the function cancelling the factors which are causing the limit to be of the form $\frac{0}{0}$.

(i) We have $\lim\limits_{x \to 1} \frac{x^2 + 1}{x + 100} = \frac{1^2 + 1}{1 + 100} = \frac{2}{101}$

(ii) Evaluating the function at 2, it is of the form $\frac{0}{0}$.

Hence $\lim\limits_{x \to 2} \frac{x^3 - 4x^2 + 4x}{x^2 - 4} = \lim\limits_{x \to 2} \frac{x(x-2)^2}{(x+2)(x-2)}$

$$= \lim_{x \to 2} \frac{x(x-2)}{(x+2)} \qquad \text{as } x \neq 2$$

$$= \frac{2(2-2)}{2+2} = \frac{0}{4} = 0.$$

(iii) Evaluating the function at 2, we get it of the form $\frac{0}{0}$.

Hence $\lim\limits_{x \to 2} \frac{x^2 - 4}{x^3 - 4x^2 + 4x} = \lim\limits_{x \to 2} \frac{(x+2)(x-2)}{x(x-2)^2}$

$$= \lim_{x \to 2} \frac{(x+2)}{x(x-2)} = \frac{2+2}{2(2-2)} = \frac{4}{0}$$

which is not defined.

(iv) Evaluating the function at 2, we get it of the form $\frac{0}{0}$.

Hence $\lim\limits_{x \to 2} \frac{x^3 - 2x^2}{x^2 - 5x + 6} = \lim\limits_{x \to 2} \frac{x^2(x-2)}{(x-2)(x-3)}$

$$= \lim_{x \to 2} \frac{x^2}{(x-3)} = \frac{(2)^2}{2-3} = \frac{4}{-1} = -4.$$

(v) First, we rewrite the function as a rational function.

$$\left[\frac{x-2}{x^2-x}-\frac{1}{x^3-3x^2+2x}\right]=\left[\frac{x-2}{x(x-1)}-\frac{1}{x\left(x^2-3x+2\right)}\right]$$

$$=\left[\frac{x-2}{x(x-1)}-\frac{1}{x(x-1)(x-2)}\right]$$

$$=\left[\frac{x^2-4x+4-1}{x(x-1)(x-2)}\right]$$

$$=\frac{x^2-4x+3}{x(x-1)(x-2)}$$

Evaluating the function at 1, we get it of the form $\frac{0}{0}$.

Hence $$\lim_{x\to 1}\left[\frac{x^2-2}{x^2-x}-\frac{1}{x^3-3x^2+2x}\right]=\lim_{x\to 1}\frac{x^2-4x+3}{x(x-1)(x-2)}$$

$$=\lim_{x\to 1}\frac{(x-3)(x-1)}{x(x-1)(x-2)}$$

$$=\lim_{x\to 1}\frac{x-3}{x(x-2)}=\frac{1-3}{1(1-2)}=2.$$

We remark that we could cancel the term $(x - 1)$ in the above evaluation because $x \neq 1$.

Evaluation of an important limit which will be used in the sequel is given as a theorem below.

Theorem 2 For any positive integer *n*,

$$\lim_{x\to a}\frac{x^n-a^n}{x-a}=na^{n-1}.$$

Remark The expression in the above theorem for the limit is true even if n is any rational number and a is positive.

Proof Dividing $(x^n - a^n)$ by $(x - a)$, we see that

$$x^n - a^n = (x-a)\,(x^{n-1} + x^{n-2}a + x^{n-3}a^2 + ... + x\,a^{n-2} + a^{n-1})$$

Thus, $$\lim_{x\to a}\frac{x^n - a^n}{x-a} = \lim_{x\to a}(x^{n-1} + x^{n-2}a + x^{n-3}a^2 + ... + x\,a^{n-2} + a^{n-1})$$

$$= a^{n-1} + a\,a^{n-2} + . . . + a^{n-2}(a) + a^{n-1}$$

$$= a^{n-1} + a^{n-1} + ... + a^{n-1} + a^{n-1} \ (n \text{ terms})$$

$$= na^{n-1}$$

Example 3 Evaluate:

(i) $\lim_{x\to 1}\frac{x^{15}-1}{x^{10}-1}$ (ii) $\lim_{x\to 0}\frac{\sqrt{1+x}-1}{x}$

Solution (i) We have

$$\lim_{x\to 1}\frac{x^{15}-1}{x^{10}-1} = \lim_{x\to 1}\left[\frac{x^{15}-1}{x-1}\div\frac{x^{10}-1}{x-1}\right]$$

$$= \lim_{x\to 1}\left[\frac{x^{15}-1}{x-1}\right]\div\lim_{x\to 1}\left[\frac{x^{10}-1}{x-1}\right]$$

$$= 15\,(1)^{14} \div 10(1)^{9} \ \text{(by the theorem above)}$$

$$= 15 \div 10 = \frac{3}{2}$$

(ii) Put $y = 1 + x$, so that $y \to 1$ as $x \to 0$.

Then $$\lim_{x\to 0}\frac{\sqrt{1+x}-1}{x} = \lim_{y\to 1}\frac{\sqrt{y}-1}{y-1}$$

$$= \lim_{y\to 1}\frac{y^{\frac{1}{2}}-1^{\frac{1}{2}}}{y-1}$$

$$= \frac{1}{2}(1)^{\frac{1}{2}-1} \ \text{(by the remark above)} = \frac{1}{2}$$

13.4 Limits of Trigonometric Functions

The following facts (stated as theorems) about functions in general come in handy in calculating limits of some trigonometric functions.

Theorem 3 Let f and g be two real valued functions with the same domain such that $f(x) \leq g(x)$ for all x in the domain of definition, For some a, if both $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist, then $\lim_{x\to a} f(x) \leq \lim_{x\to a} g(x)$. This is illustrated in Fig 13.8.

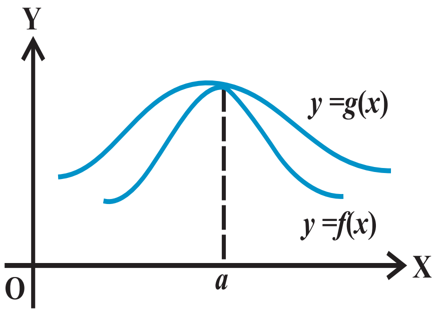

Fig 13.8

Theorem 4 (Sandwich Theorem) Let f, g and h be real functions such that $f(x) \leq g(x) \leq h(x)$ for all x in the common domain of definition. For some real number a, if $\lim_{x\to a} f(x) = l = \lim_{x\to a} h(x)$, then $\lim_{x\to a} g(x) = l$. This is illustrated in Fig 13.9.

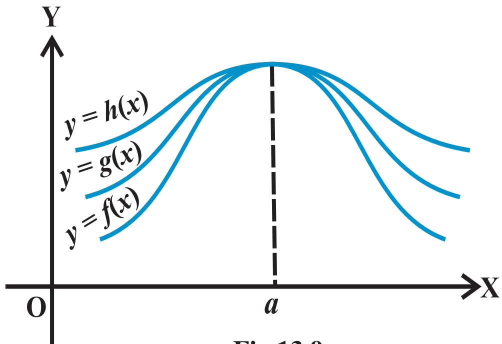

Fig 13.9

Given below is a beautiful geometric proof of the following important inequality relating trigonometric functions.

$$\cos x < \frac{\sin x}{x} < 1 \quad \text{for } 0 < |x| < \frac{\pi}{2} \qquad (*)$$

Proof We know that sin (– x) = – sin x and cos(– x) = cos x. Hence, it is sufficient to prove the inequality for $0 < x < \frac{\pi}{2}$.

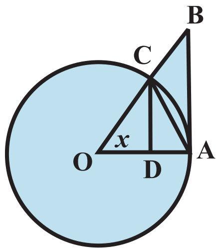

Fig 13.10

In the Fig 13.10, O is the centre of the unit circle such that the angle AOC is x radians and $0 < x < \frac{\pi}{2}$. Line segments B A and CD are perpendiculars to OA. Further, join AC. Then

Area of ΔOAC < Area of sector OAC < Area of Δ OAB.

i.e., $\frac{1}{2}OA.CD < \frac{x}{2\pi}.\pi.(OA)^2 < \frac{1}{2}OA.AB$.

i.e., CD < x . OA < AB.

From Δ OCD,

$\sin x = \frac{CD}{OA}$ (since OC = OA) and hence CD = OA sin x. Also $\tan x = \frac{AB}{OA}$ and

hence AB = OA. tan x. Thus

OA sin x < OA. x < OA. tan x.

Since length OA is positive, we have

sin x < x < tan x.

Since $0 < x < \frac{\pi}{2}$, sin$x$ is positive and thus by dividing throughout by sin x, we have

$1 < \frac{x}{\sin x} < \frac{1}{\cos x}$. Taking reciprocals throughout, we have

$\cos x < \frac{\sin x}{x} < 1$

which complete the proof.

Theorem 5 The following are two important limits.

(i) $\lim_{x \to 0} \frac{\sin x}{x} = 1$. (ii) $\lim_{x \to 0} \frac{1 - \cos x}{x} = 0$.

Proof (i) The inequality in (*) says that the function $\frac{\sin x}{x}$ is sandwiched between the function cos x and the constant function which takes value 1.

Further, since $\lim_{x\to 0} \cos x = 1$, we see that the proof of (i) of the theorem is complete by sandwich theorem.

To prove (ii), we recall the trigonometric identity $1 - \cos x = 2 \sin^2\left(\frac{x}{2}\right)$.

Then

$$\lim_{x\to 0}\frac{1-\cos x}{x} = \lim_{x\to 0}\frac{2\sin^2\left(\frac{x}{2}\right)}{x} = \lim_{x\to 0}\frac{\sin\left(\frac{x}{2}\right)}{\frac{x}{2}}.\sin\left(\frac{x}{2}\right)$$

$$= \lim_{x\to 0}\frac{\sin\left(\frac{x}{2}\right)}{\frac{x}{2}}.\lim_{x\to 0}\sin\left(\frac{x}{2}\right) = 1.0 = 0$$

Observe that we have implicitly used the fact that $x \to 0$ is equivalent to $\frac{x}{2} \to 0$. This may be justified by putting $y = \frac{x}{2}$.

Example 4 Evaluate: (i) $\lim_{x\to 0}\frac{\sin 4x}{\sin 2x}$ (ii) $\lim_{x\to 0}\frac{\tan x}{x}$

Solution (i) $\lim_{x\to 0}\frac{\sin 4x}{\sin 2x} = \lim_{x\to 0}\left[\frac{\sin 4x}{4x}.\frac{2x}{\sin 2x}.2\right]$

$$= 2.\lim_{x\to 0}\left[\frac{\sin 4x}{4x}\right] \div \left[\frac{\sin 2x}{2x}\right]$$

$$= 2.\lim_{4x\to 0}\left[\frac{\sin 4x}{4x}\right] \div \lim_{2x\to 0}\left[\frac{\sin 2x}{2x}\right]$$

$= 2.1.1 = 2$ (as $x \to 0$, $4x \to 0$ and $2x \to 0$)

(ii) We have $\lim\limits_{x\to 0}\frac{\tan x}{x} = \lim\limits_{x\to 0}\frac{\sin x}{x\cos x} = \lim\limits_{x\to 0}\frac{\sin x}{x}\cdot\lim\limits_{x\to 0}\frac{1}{\cos x} = 1.1 = 1$

A general rule that needs to be kept in mind while evaluating limits is the following.

Say, given that the limit $\lim\limits_{x\to a}\frac{f(x)}{g(x)}$ exists and we want to evaluate this. First we check the value of $f(a)$ and $g(a)$. If both are 0, then we see if we can get the factor which is causing the terms to vanish, i.e., see if we can write $f(x) = f_1(x)\, f_2(x)$ so that $f_1(a) = 0$ and $f_2(a) \neq 0$. Similarly, we write $g(x) = g_1(x)\, g_2(x)$, where $g_1(a) = 0$ and $g_2(a) \neq 0$. Cancel out the common factors from $f(x)$ and $g(x)$ (if possible) and write

$$\frac{f(x)}{g(x)} = \frac{p(x)}{q(x)},\ \text{where } q(x) \neq 0.$$

Then $$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{p(a)}{q(a)}.$$

EXERCISE 13.1

Evaluate the following limits in Exercises 1 to 22.

1. $\lim\limits_{x\to 3} x+3$

2. $\lim\limits_{x\to \pi}\left(x-\frac{22}{7}\right)$

3. $\lim\limits_{r\to 1}\pi r^2$

4. $\lim\limits_{x\to 4}\frac{4x+3}{x-2}$

5. $\lim\limits_{x\to -1}\frac{x^{10}+x^5+1}{x-1}$

6. $\lim\limits_{x\to 0}\frac{(x+1)^5-1}{x}$

7. $\lim\limits_{x\to 2}\frac{3x^2-x-10}{x^2-4}$

8. $\lim\limits_{x\to 3}\frac{x^4-81}{2x^2-5x-3}$

9. $\lim\limits_{x\to 0}\frac{ax+b}{cx+1}$

10. $\lim\limits_{z\to 1}\frac{z^{\frac{1}{3}}-1}{z^{\frac{1}{6}}-1}$

11. $\lim\limits_{x\to 1}\frac{ax^2+bx+c}{cx^2+bx+a},\ a+b+c\neq 0$

12. $\lim\limits_{x\to -2}\frac{\frac{1}{x}+\frac{1}{2}}{x+2}$

13. $\lim\limits_{x\to 0}\frac{\sin ax}{bx}$

14. $\lim\limits_{x\to 0}\frac{\sin ax}{\sin bx},\ a,b\neq 0$

15. $\lim_{x \to \pi} \frac{\sin(\pi - x)}{\pi(\pi - x)}$

16. $\lim_{x \to 0} \frac{\cos x}{\pi - x}$

17. $\lim_{x \to 0} \frac{\cos 2x - 1}{\cos x - 1}$

18. $\lim_{x \to 0} \frac{ax + x\cos x}{b\sin x}$

19. $\lim_{x \to 0} x\sec x$

20. $\lim_{x \to 0} \frac{\sin ax + bx}{ax + \sin bx}\, a, b, a + b \neq 0,$

21. $\lim_{x \to 0} (\operatorname{cosec} x - \cot x)$

22. $\lim_{x \to \frac{\pi}{2}} \frac{\tan 2x}{x - \frac{\pi}{2}}$

23. Find $\lim_{x \to 0} f(x)$ and $\lim_{x \to 1} f(x)$, where $f(x) = \begin{cases} 2x + 3, & x \leq 0 \\ 3(x+1), & x > 0 \end{cases}$

24. Find $\lim_{x \to 1} f(x)$, where $f(x) = \begin{cases} x^2 - 1, & x \leq 1 \\ -x^2 - 1, & x > 1 \end{cases}$

25. Evaluate $\lim_{x \to 0} f(x)$, where $f(x) = \begin{cases} \frac{|x|}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

26. Find $\lim_{x \to 0} f(x)$, where $f(x) = \begin{cases} \frac{x}{|x|}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

27. Find $\lim_{x \to 5} f(x)$, where $f(x) = |x| - 5$

28. Suppose $f(x) = \begin{cases} a + bx, & x < 1 \\ 4, & x = 1 \\ b - ax, & x > 1 \end{cases}$

and if $\lim_{x \to 1} f(x) = f(1)$ what are possible values of a and b?

29. Let $a_1, a_2, \ldots, a_n$ be fixed real numbers and define a function

$$f(x) = (x - a_1)(x - a_2) \ldots (x - a_n).$$

What is $\lim_{x \to a_1} f(x)$? For some $a \neq a_1, a_2, \ldots, a_n$, compute $\lim_{x \to a} f(x)$.

30. If $f(x) = \begin{cases} |x| + 1, & x < 0 \\ 0, & x = 0 \\ |x| - 1, & x > 0 \end{cases}$.

For what value (s) of a does $\lim_{x \to a} f(x)$ exists?

31. If the function $f(x)$ satisfies $\lim_{x \to 1} \frac{f(x) - 2}{x^2 - 1} = \pi$, evaluate $\lim_{x \to 1} f(x)$.

32. If $f(x) = \begin{cases} mx^2 + n, & x < 0 \\ nx + m, & 0 \le x \le 1 \\ nx^3 + m, & x > 1 \end{cases}$. For what integers m and n does both $\lim_{x \to 0} f(x)$ and $\lim_{x \to 1} f(x)$ exist?

13.5 Derivatives

We have seen in the Section 13.2, that by knowing the position of a body at various time intervals it is possible to find the rate at which the position of the body is changing. It is of very general interest to know a certain parameter at various instants of time and try to finding the rate at which it is changing. There are several real life situations where such a process needs to be carried out. For instance, people maintaining a reservoir need to know when will a reservoir overflow knowing the depth of the water at several instances of time, Rocket Scientists need to compute the precise velocity with which the satellite needs to be shot out from the rocket knowing the height of the rocket at various times. Financial institutions need to predict the changes in the value of a particular stock knowing its present value. In these, and many such cases it is desirable to know how a particular parameter is changing with respect to some other parameter. The heart of the matter is derivative of a function at a given point in its domain of definition.