17. A man starts repaying a loan as first instalment of Rs. 100. If he increases the instalment by Rs 5 every month, what amount he will pay in the 30th instalment?

18. The difference between any two consecutive interior angles of a polygon is 5°. If the smallest angle is 120°, find the number of the sides of the polygon.

9.5 Geometric Progression (G. P.)

Let us consider the following sequences:

$$\text{(i) } 2,4,8,16,\ldots, \text{ (ii) } \frac{1}{9},\frac{-1}{27},\frac{1}{81},\frac{-1}{243}\ldots \text{ (iii) } .01,.0001,.000001,\ldots$$

In each of these sequences, how their terms progress? We note that each term, except the first progresses in a definite order.

In (i), we have $a_1 = 2, \frac{a_2}{a_1} = 2, \frac{a_3}{a_2} = 2, \frac{a_4}{a_3} = 2$ and so on.

In (ii), we observe, $a_1 = \frac{1}{9}, \frac{a_2}{a_1} = -\frac{1}{3}, \frac{a_3}{a_2} = -\frac{1}{3}, \frac{a_4}{a_3} = -\frac{1}{3}$ and so on.

Similarly, state how do the terms in (iii) progress? It is observed that in each case, every term except the first term bears a constant ratio to the term immediately preceding it. In (i), this constant ratio is 2; in (ii), it is $-\frac{1}{3}$ and in (iii), the constant ratio is 0.01. Such sequences are called *geometric sequence* or *geometric progression* abbreviated as G.P.

A sequence $a_1, a_2, a_3, \ldots, a_n, \ldots$ is called *geometric progression*, if each term is non-zero and $\frac{a_{k+1}}{a_k} = r$ (constant), for $k \geq 1$.

By letting $a_1 = a$, we obtain a geometric progression, $a, ar, ar^2, ar^3, \ldots.,$ where a is called the *first term* and r is called the *common ratio* of the G.P. Common ratio in geometric progression (i), (ii) and (iii) above are 2, $-\frac{1}{3}$ and 0.01, respectively.

As in case of arithmetic progression, the problem of finding the n^{th} term or sum of n terms of a geometric progression containing a large number of terms would be difficult without the use of the formulae which we shall develop in the next Section. We shall use the following notations with these formulae:

a = the first term, r = the common ratio, l = the last term,

n = the numbers of terms,

n = the numbers of terms,

S_n = the sum of first n terms.

9.5.1 General term of a G.P. Let us consider a G.P. with first non-zero term 'a' and common ratio 'r'. Write a few terms of it. The second term is obtained by multiplying a by r, thus $a_2 = ar$. Similarly, third term is obtained by multiplying a_2 by r. Thus, $a_3 = a_2 r = ar^2$, and so on.

We write below these and few more terms.

1[st] term = $a_1 = a = ar^{1-1}$, 2[nd] term = $a_2 = ar = ar^{2-1}$, 3[rd] term = $a_3 = ar^2 = ar^{3-1}$
4[th] term = $a_4 = ar^3 = ar^{4-1}$, 5[th] term = $a_5 = ar^4 = ar^{5-1}$
Do you see a pattern? What will be 16[th] term?

$$a_{16} = ar^{16-1} = ar^{15}$$

Therefore, the pattern suggests that the n^{th} term of a G.P. is given by

$a_n = ar^{n-1}$.

Thus, a, G.P. can be written as $a, ar, ar^2, ar^3, \ldots ar^{n-1}$; $a, ar, ar^2, \ldots, ar^{n-1} \ldots$;according as G.P. is *finite* or *infinite*, respectively.

The series $a + ar + ar^2 + \ldots + ar^{n-1}$ or $a + ar + ar^2 + \ldots + ar^{n-1} + \ldots$ are called *finite* or *infinite geometric series*, respectively.

9.5.2. Sum to n terms of a G.P. Let the first term of a G.P. be a and the common ratio be r. Let us denote by S_n the sum to first n terms of G.P. Then

$$S_n = a + ar + ar^2 + \ldots + ar^{n-1} \quad \ldots (1)$$

Case 1 If $r = 1$, we have $S_n = a + a + a + \ldots + a$ (n terms) $= na$

Case 2 If $r \neq 1$, multiplying (1) by r, we have

$$rS_n = ar + ar^2 + ar^3 + \ldots + ar^n \quad \ldots (2)$$

Subtracting (2) from (1), we get $(1 - r)\, S_n = a - ar^n = a(1 - r^n)$

This gives $S_n = \dfrac{a(1-r^n)}{1-r}$ or $S_n = \dfrac{a(r^n-1)}{r-1}$

Example 9 Find the 10[th] and n^{th} terms of the G.P. 5, 25,125,… .

Solution Here $a = 5$ and $r = 5$. Thus, $a_{10} = 5(5)^{10-1} = 5(5)^9 = 5^{10}$
and $a_n = ar^{n-1} = 5(5)^{n-1} = 5^n$.

Example10 Which term of the G.P., 2,8,32, ... up to n terms is 131072?

Solution Let 131072 be the n^{th} term of the given G.P. Here $a = 2$ and $r = 4$.
Therefore $131072 = a_n = 2(4)^{n-1}$ or $65536 = 4^{n-1}$

This gives $4^8 = 4^{n-1}$.

So that $n - 1 = 8$, i.e., $n = 9$. Hence, 131072 is the 9[th] term of the G.P.

Example11 In a G.P., the 3rd term is 24 and the 6th term is 192.Find the 10th term.

Solution Here, $a_3 = ar^2 = 24$ ... (1)

and $a_6 = ar^5 = 192$ ... (2)

Dividing (2) by (1), we get $r = 2$. Substituting $r = 2$ in (1), we get $a = 6$.

Hence $a_{10} = 6\,(2)^9 = 3072$.

Example12 Find the sum of first n terms and the sum of first 5 terms of the geometric series $1+\frac{2}{3}+\frac{4}{9}+...$

Solution Here $a = 1$ and $r = \frac{2}{3}$. Therefore

$$S_n = \frac{a\,(1-r^n)}{1-r} = \frac{\left[1-\left(\frac{2}{3}\right)^n\right]}{1-\frac{2}{3}} = 3\left[1-\left(\frac{2}{3}\right)^n\right]$$

In particular, $S_5 = 3\left[1-\left(\frac{2}{3}\right)^5\right] = 3\times\frac{211}{243} = \frac{211}{81}$.

Example 13 How many terms of the G.P. $3, \frac{3}{2}, \frac{3}{4}, ...$ are needed to give the sum $\frac{3069}{512}$?

Solution Let n be the number of terms needed. Given that $a = 3$, $r = \frac{1}{2}$ and $S_n = \frac{3069}{512}$

Since $$S_n = \frac{a\,(1-r^n)}{1-r}$$

Therefore $$\frac{3069}{512} = \frac{3(1-\frac{1}{2^n})}{1-\frac{1}{2}} = 6\left(1-\frac{1}{2^n}\right)$$

or $$\frac{3069}{3072} = 1-\frac{1}{2^n}$$

or $$\frac{1}{2^n} = 1-\frac{3069}{3072} = \frac{3}{3072} = \frac{1}{1024}$$

or $2^n = 1024 = 2^{10}$, which gives $n = 10$.

Example 14 The sum of first three terms of a G.P. is $\frac{13}{12}$ and their product is – 1. Find the common ratio and the terms.

Solution Let $\frac{a}{r}$, a, ar be the first three terms of the G.P. Then

$$\frac{a}{r} + ar + a = \frac{13}{12} \qquad \text{... (1)}$$

and $$\left(\frac{a}{r}\right)(a)\,(ar) = -1 \qquad \text{... (2)}$$

From (2), we get $a^3 = -1$, i.e., $a = -1$ (considering only real roots)

Substituting $a = -1$ in (1), we have

$$-\frac{1}{r} - 1 - r = \frac{13}{12} \text{ or } 12r^2 + 25r + 12 = 0.$$

This is a quadratic in r, solving, we get $r = -\frac{3}{4}$ or $-\frac{4}{3}$.

Thus, the three terms of G.P. are : $\frac{4}{3}, -1, \frac{3}{4}$ for $r = \frac{-3}{4}$ and $\frac{3}{4}, -1, \frac{4}{3}$ for $r = \frac{-4}{3}$,

Example15 Find the sum of the sequence 7, 77, 777, 7777, ... to n terms.

Solution This is not a G.P., however, we can relate it to a G.P. by writing the terms as

$$S_n = 7 + 77 + 777 + 7777 + \ldots \text{ to } n \text{ terms}$$

$$= \frac{7}{9}[9+99+999+9999+ \ldots\text{to } n \text{ term}]$$

$$= \frac{7}{9}[(10-1)+(10^2-1)+(10^3-1)+(10^4-1)+\ldots n \text{ terms}]$$

$$= \frac{7}{9}[(10+10^2+10^3+...n\text{ terms})-(1+1+1+...n\text{ terms})]$$

$$= \frac{7}{9}\left[\frac{10(10^n-1)}{10-1}-n\right] = \frac{7}{9}\left[\frac{10\,(10^n-1)}{9}-n\right].$$

Example 16 A person has 2 parents, 4 grandparents, 8 great grandparents, and so on. Find the number of his ancestors during the ten generations preceding his own.

Solution Here $a = 2$, $r = 2$ and $n = 10$

Using the sum formula $S_n = \dfrac{a\,(r^n-1)}{r-1}$

We have $S_{10} = 2(2^{10} - 1) = 2046$

Hence, the number of ancestors preceding the person is 2046.

9.5.3 *Geometric Mean (G.M.)* The geometric mean of two positive numbers a and b is the number $\sqrt{ab}$. Therefore, the geometric mean of 2 and 8 is 4. We observe that the three numbers 2,4,8 are consecutive terms of a G.P. This leads to a generalisation of the concept of geometric means of two numbers.

Given any two positive numbers a and b, we can insert as many numbers as we like between them to make the resulting sequence in a G.P.

Let $G_1, G_2,\ldots, G_n$ be n numbers between positive numbers a and b such that $a,G_1,G_2,G_3,\ldots,G_n,b$ is a G.P. Thus, b being the $(n + 2)^{th}$ term,we have

$$b = ar^{n+1}, \quad \text{or} \quad r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}.$$

Hence $$G_1 = ar = a\left(\frac{b}{a}\right)^{\frac{1}{n+1}}, \quad G_2 = ar^2 = a\left(\frac{b}{a}\right)^{\frac{2}{n+1}}, \quad G_3 = ar^3 = a\left(\frac{b}{a}\right)^{\frac{3}{n+1}},$$

$$G_n = ar^n = a\left(\frac{b}{a}\right)^{\frac{n}{n+1}}$$

Example17 Insert three numbers between 1 and 256 so that the resulting sequence is a G.P.

Solution Let G_1, G_2,G_3 be three numbers between 1 and 256 such that $1, G_1,G_2,G_3, 256$ is a G.P.

Therefore $256 = r^4$ giving $r = \pm 4$ (Taking real roots only)

For $r = 4$, we have $G_1 = ar = 4$, $G_2 = ar^2 = 16$, $G_3 = ar^3 = 64$

Similarly, for $r = -4$, numbers are $-4, 16$ and -64.

Hence, we can insert 4, 16, 64 between 1 and 256 so that the resulting sequences are in G.P.

9.6 Relationship Between A.M. and G.M.

Let A and G be A.M. and G.M. of two given positive real numbers a and b, respectively. Then

$$A = \frac{a+b}{2} \text{ and } G = \sqrt{ab}$$

Thus, we have

$$A - G = \frac{a+b}{2} - \sqrt{ab} = \frac{a+b-2\sqrt{ab}}{2}$$

$$= \frac{\left(\sqrt{a} - \sqrt{b}\right)^2}{2} \geq 0 \qquad \ldots (1)$$

From (1), we obtain the relationship $A \geq G$.

Example 18 If A.M. and G.M. of two positive numbers a and b are 10 and 8, respectively, find the numbers.

Solution Given that $\text{A.M.} = \frac{a+b}{2} = 10$ ... (1)

and $\text{G.M.} = \sqrt{ab} = 8$ ... (2)

From (1) and (2), we get

$$a + b = 20 \qquad \ldots (3)$$

$$ab = 64 \qquad \ldots (4)$$

Putting the value of a and b from (3), (4) in the identity $(a - b)^2 = (a + b)^2 - 4ab$, we get

$$(a - b)^2 = 400 - 256 = 144$$

or $a - b = \pm 12$

... (5)

Solving (3) and (5), we obtain

$$a = 4, b = 16 \text{ or } a = 16, b = 4$$

Thus, the numbers a and b are 4, 16 or 16, 4 respectively.

EXERCISE 9.3

1. Find the 20^{th} and n^{th} terms of the G.P. $\frac{5}{2}, \frac{5}{4}, \frac{5}{8}, ...$

2. Find the 12^{th} term of a G.P. whose 8^{th} term is 192 and the common ratio is 2.

3. The 5^{th}, 8^{th} and 11^{th} terms of a G.P. are p, q and s, respectively. Show that $q^2 = ps$.

4. The 4^{th} term of a G.P. is square of its second term, and the first term is – 3. Determine its 7^{th} term.

5. Which term of the following sequences:

(a) $2, 2\sqrt{2}, 4, ...$ is 128 ? (b) $\sqrt{3}, 3, 3\sqrt{3}, ...$ is 729 ?

(c) $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, ...$ is $\frac{1}{19683}$?

6. For what values of x, the numbers $-\frac{2}{7}, x, -\frac{7}{2}$ are in G.P.?

Find the sum to indicated number of terms in each of the geometric progressions in Exercises 7 to 10:

7. 0.15, 0.015, 0.0015, ... 20 terms.

8. $\sqrt{7}, \sqrt{21}, 3\sqrt{7}, ...$ n terms.

9. $1, -a, a^2, -a^3, ...$ n terms (if $a \neq -1$).

10. $x^3, x^5, x^7, ...$ n terms (if $x \neq \pm 1$).

11. Evaluate $\sum_{k=1}^{11} (2 + 3^k)$.

12. The sum of first three terms of a G.P. is $\frac{39}{10}$ and their product is 1. Find the common ratio and the terms.

13. How many terms of G.P. $3, 3^2, 3^3, …$ are needed to give the sum 120?

14. The sum of first three terms of a G.P. is 16 and the sum of the next three terms is 128. Determine the first term, the common ratio and the sum to n terms of the G.P.

15. Given a G.P. with $a = 729$ and 7^{th} term 64, determine S_7.

16. Find a G.P. for which sum of the first two terms is – 4 and the fifth term is 4 times the third term.

17. If the 4^{th}, 10^{th} and 16^{th} terms of a G.P. are x, y and z, respectively. Prove that x, y, z are in G.P.

18. Find the sum to n terms of the sequence, 8, 88, 888, 8888… .

19. Find the sum of the products of the corresponding terms of the sequences 2, 4, 8, 16, 32 and 128, 32, 8, 2, $\frac{1}{2}$.

20. Show that the products of the corresponding terms of the sequences $a, ar, ar^2, \ldots ar^{n-1}$ and A, AR, AR^2, … AR^{n-1} form a G.P, and find the common ratio.

21. Find four numbers forming a geometric progression in which the third term is greater than the first term by 9, and the second term is greater than the 4th by 18.

22. If the p^{th}, q^{th} and r^{th} terms of a G.P. are a, b and c, respectively. Prove that

$$a^{q-r}\, b^{r-p} c^{P-q} = 1.$$

23. If the first and the n^{th} term of a G.P. are a and b, respectively, and if P is the product of n terms, prove that $P^2 = (ab)^n$.

24. Show that the ratio of the sum of first n terms of a G.P. to the sum of terms from $(n + 1)^{th}$ to $(2n)^{th}$ term is $\frac{1}{r^n}$.

25. If a, b, c and d are in G.P. show that

$(a^2 + b^2 + c^2)(b^2 + c^2 + d^2) = (ab + bc + cd)^2$.

26. Insert two numbers between 3 and 81 so that the resulting sequence is G.P.

27. Find the value of n so that $\frac{a^{n+1}+b^{n+1}}{a^n+b^n}$ may be the geometric mean between a and b.

28. The sum of two numbers is 6 times their geometric mean, show that numbers are in the ratio $\left(3+2\sqrt{2}\right):\left(3-2\sqrt{2}\right)$.

29. If A and G be A.M. and G.M., respectively between two positive numbers, prove that the numbers are $A \pm \sqrt{(A+G)(A-G)}$.

30. The number of bacteria in a certain culture doubles every hour. If there were 30 bacteria present in the culture originally, how many bacteria will be present at the end of 2nd hour, 4th hour and n^{th} hour ?

31. What will Rs 500 amounts to in 10 years after its deposit in a bank which pays annual interest rate of 10% compounded annually?

32. If A.M. and G.M. of roots of a quadratic equation are 8 and 5, respectively, then obtain the quadratic equation.