Let $a_1, a_2, a_3 \cdots$ be a sequence of positive integers in arithmetic progression with common difference 2. Also, let $b_1, b_2, b_3, \cdots$ be a sequence of positive integers in G.P with common ratio 2. If $a_1 = b_1 = c$, then the number of all possible values of c, for which the equality $2(a_1 + a_2 + \cdots + a_n) = b_1 + b_2 + \cdots + b_n$ holds for some positive integer n, is ______

(JEE Advanced 2020)

Solution:

$\because 2(a_1 + a_2 + \cdots + a_n) = b_1 + b_2 + \cdots + b_n$

$\Rightarrow 2\left(\frac{n}{2}\left[2c + (n-1)2\right]\right) = c\left(\frac{2^n - 1}{2 - 1}\right)$

$\Rightarrow 2nc + 2n^2 - 2n = 2^n c - c$

$\because c \in N \Rightarrow 2n^2 - 2n \geq 2^n - 2n - 1$

$\Rightarrow 2n^2 + 1 \geq 2^n \Rightarrow n \leq 6$

also, $c > 0 \Rightarrow n > 2$

$\therefore$ The possible values of n are $3, 4, 5, 6$

at $n = 3$, $c = \dfrac{2n^2 - 2n}{2^n - 1 - 2n} = \dfrac{2 \times 9 - 6}{8 - 1 - 6} = 12$

at $n = 4$, $c = \dfrac{32 - 8}{16 - 1 - 8} = \dfrac{24}{9} = \dfrac{8}{3} \notin N$

at $n = 5$, $c = \dfrac{50 - 10}{32 - 1 - 10} = \dfrac{40}{21} \notin N$

at $n = 6$, $c = \dfrac{72 - 12}{64 - 1 - 12} = \dfrac{60}{51} \notin N$

$\therefore$ Required value of $c = 12$ for $n = 3$