9. $x\dfrac{dy}{dx} + y - x + xy \cot x = 0 \ \ (x \neq 0)$ **10.** $(x + y)\dfrac{dy}{dx} = 1$

11. $y\, dx + (x - y^2)\, dy = 0$ **12.** $(x + 3y^2)\dfrac{dy}{dx} = y \ \ (y > 0)$.

For each of the differential equations given in Exercises 13 to 15, find a particular solution satisfying the given condition:

13. $\dfrac{dy}{dx} + 2y \tan x = \sin x;\ y = 0$ when $x = \dfrac{\pi}{3}$

14. $(1 + x^2)\dfrac{dy}{dx} + 2xy = \dfrac{1}{1 + x^2};\ y = 0$ when $x = 1$

15. $\dfrac{dy}{dx} - 3y \cot x = \sin 2x;\ y = 2$ when $x = \dfrac{\pi}{2}$

16. Find the equation of a curve passing through the origin given that the slope of the tangent to the curve at any point (x, y) is equal to the sum of the coordinates of the point.

17. Find the equation of a curve passing through the point (0, 2) given that the sum of the coordinates of any point on the curve exceeds the magnitude of the slope of the tangent to the curve at that point by 5.

18. The Integrating Factor of the differential equation $x\dfrac{dy}{dx} - y = 2x^2$ is

(A) e^{-x} (B) e^{-y} (C) $\dfrac{1}{x}$ (D) x

19. The Integrating Factor of the differential equation

$$(1 - y^2)\frac{dx}{dy} + yx = ay(\ 1 \quad y \quad 1) \text{ is}$$

(A) $\dfrac{1}{y^2 \quad 1}$ (B) $\dfrac{1}{\sqrt{y^2 - 1}}$ (C) $\dfrac{1}{1 - y^2}$ (D) $\dfrac{1}{\sqrt{1 - y^2}}$

Miscellaneous Examples

Example 24 Verify that the function $y = c_1 e^{ax} \cos bx + c_2 e^{ax} \sin bx$, where c_1, c_2 are arbitrary constants is a solution of the differential equation

$$\frac{d^2y}{dx^2} - 2a\frac{dy}{dx} + \left(a^2 + b^2\right)y = 0$$

Solution The given function is

$$y = e^{ax}\,[c_1 \cos bx + c_2 \sin bx] \qquad \ldots (1)$$

Differentiating both sides of equation (1) with respect to x, we get

$$\frac{dy}{dx} = e^{ax} \quad -bc_1 \sin bx \quad bc_2 \cos bx \qquad c_1 \cos bx \quad c_2 \sin bx \;\; e^{ax} \quad a$$

or

$$\frac{dy}{dx} = e^{ax}[(bc_2 + ac_1)\cos bx + (ac_2 - bc_1)\sin bx] \qquad \ldots (2)$$

Differentiating both sides of equation (2) with respect to x, we get

$$\frac{d^2y}{dx^2} = e^{ax}[(bc_2 \quad ac_1)(\quad b\sin bx) \quad (ac_2 \quad bc_1)(b\cos bx)]$$

$$+ [(bc_2 + ac_1)\cos bx + (ac_2 - bc_1)\sin bx]\,e^{ax}.a$$

$$= e^{ax}[(a^2 c_2 - 2abc_1 - b^2 c_2)\sin bx + (a^2 c_1 + 2abc_2 - b^2 c_1)\cos bx]$$

Substituting the values of $\frac{d^2y}{dx^2}, \frac{dy}{dx}$ and y in the given differential equation, we get

$$\text{L.H.S.} = e^{ax}[a^2c_2 - 2abc_1 - b^2c_2)\sin bx + (a^2c_1 + 2abc_2 - b^2c_1)\cos bx]$$

$$2ae^{ax}[(bc_2 \quad ac_1)\cos bx \quad (ac_2 \quad bc_1)\sin bx]$$

$$(a^2 \quad b^2)e^{ax}[c_1 \cos bx \quad c_2 \sin bx]$$

$$= e^{ax}\begin{bmatrix} \left(a^2c_2 - 2abc_1 - b^2c_2 - 2a^2c_2 + 2abc_1 + a^2c_2 + b^2c_2\right)\sin bx \\ +(a^2c_1 + 2abc_2 - b^2c_1 - 2abc_2 - 2a^2c_1 + a^2c_1 + b^2c_1)\cos bx \end{bmatrix}$$

$$= e^{ax}[0 \times \sin bx + 0 \cos bx] = e^{ax} \times 0 = 0 \;\; = \text{R.H.S.}$$

Hence, the given function is a solution of the given differential equation.

Example 25 Form the differential equation of the family of circles in the second quadrant and touching the coordinate axes.

Solution Let C denote the family of circles in the second quadrant and touching the coordinate axes. Let $(-a, a)$ be the coordinate of the centre of any member of this family (see Fig 9.6).

Equation representing the family C is

$$(x + a)^2 + (y - a)^2 = a^2 \qquad \text{... (1)}$$

or $$x^2 + y^2 + 2ax - 2ay + a^2 = 0 \qquad \text{... (2)}$$

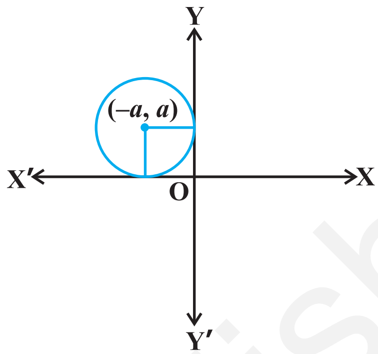

Fig 9.6

Differentiating equation (2) with respect to x, we get

$$2x + 2y\frac{dy}{dx} + 2a - 2a\frac{dy}{dx} = 0$$

or $$x + y\frac{dy}{dx} = a\left(\frac{dy}{dx} - 1\right)$$

or $$a = \frac{x + y\,y'}{y' - 1}$$

Substituting the value of a in equation (1), we get

$$\left[x + \frac{x + y\,y'}{y' - 1}\right]^2 + \left[y - \frac{x + y\,y'}{y' - 1}\right]^2 = \left[\frac{x + y\,y'}{y' - 1}\right]^2$$

or $$[xy' - x + x + y\,y']^2 + [y\,y' - y - x - y\,y']^2 = [x + y\,y']^2$$

or $$(x + y)^2\, y'^2 + [x + y]^2 = [x + y\,y']^2$$

or $$(x + y)^2\, [(y')^2 + 1] = [x + y\,y']^2$$

which is the differential equation representing the given family of circles.

Example 26 Find the particular solution of the differential equation $\log\left(\frac{dy}{dx}\right) = 3x + 4y$ given that $y = 0$ when $x = 0$.

Solution The given differential equation can be written as

$$\frac{dy}{dx} = e^{(3x + 4y)}$$

or $$\frac{dy}{dx} = e^{3x} \cdot e^{4y} \qquad \text{... (1)}$$

Separating the variables, we get

$$\frac{dy}{e^{4y}} = e^{3x}\, dx$$

Therefore $$\int e^{-4y}\,dy = \int e^{3x}\,dx$$

or $$\frac{e^{-4y}}{-4} = \frac{e^{3x}}{3} + C$$

or $$4\,e^{3x} + 3\,e^{-4y} + 12\,C = 0 \qquad \text{... (2)}$$

Substituting $x = 0$ and $y = 0$ in (2), we get

$$4 + 3 + 12\,C = 0 \text{ or } C = \frac{-7}{12}$$

Substituting the value of C in equation (2), we get

$$4e^{3x} + 3e^{-4y} - 7 = 0,$$

which is a particular solution of the given differential equation.

Example 27 Solve the differential equation

$$(x\,dy - y\,dx)\,y\sin\left(\frac{y}{x}\right) = (y\,dx + x\,dy)\,x\cos\left(\frac{y}{x}\right).$$

Solution The given differential equation can be written as

$$\left[x\,y\sin\left(\frac{y}{x}\right) - x^2\cos\left(\frac{y}{x}\right)\right]dy = \left[xy\cos\left(\frac{y}{x}\right) + y^2\sin\left(\frac{y}{x}\right)\right]dx$$

or $$\frac{dy}{dx} = \frac{xy\cos\left(\frac{y}{x}\right) + y^2\sin\left(\frac{y}{x}\right)}{xy\sin\left(\frac{y}{x}\right) - x^2\cos\left(\frac{y}{x}\right)}$$

Dividing numerator and denominator on RHS by x^2, we get

$$\frac{dy}{dx} = \frac{\frac{y}{x}\cos\left(\frac{y}{x}\right) + \left(\frac{y^2}{x^2}\right)\sin\left(\frac{y}{x}\right)}{\frac{y}{x}\sin\left(\frac{y}{x}\right) - \cos\left(\frac{y}{x}\right)} \qquad \text{... (1)}$$

Clearly, equation (1) is a homogeneous differential equation of the form $\frac{dy}{dx} = g\left(\frac{y}{x}\right)$.

To solve it, we make the substitution

$$y = vx \qquad \text{... (2)}$$

or $$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

or $$v + x\frac{dv}{dx} = \frac{v\cos v + v^2 \sin v}{v\sin v - \cos v} \qquad \text{(using (1) and (2))}$$

or $$x\frac{dv}{dx} = \frac{2v\cos v}{v\sin v - \cos v}$$

or $$\left(\frac{v\sin v - \cos v}{v\cos v}\right) dv = \frac{2\,dx}{x}$$

Therefore $$\int \left(\frac{v\sin v - \cos v}{v\cos v}\right) dv = 2\int \frac{1}{x}\,dx$$

or $$\int \tan v\, dv - \int \frac{1}{v}\, dv = 2\int \frac{1}{x}\,dx$$

or $$\log|\sec v| - \log|v| = 2\log|x| + \log|C_1|$$

or $$\log\left|\frac{\sec v}{v\,x^2}\right| = \log|C_1|$$

or $$\frac{\sec v}{v\,x^2} = \pm C_1 \qquad \text{... (3)}$$

Replacing v by $\frac{y}{x}$ in equation (3), we get

$$\frac{\sec\left(\frac{y}{x}\right)}{\left(\frac{y}{x}\right)(x^2)} = C \text{ where, } C = \pm C_1$$

or $$\sec\left(\frac{y}{x}\right) = C\,xy$$

which is the general solution of the given differential equation.

Example 28 Solve the differential equation

$$(\tan^{-1}y - x)\, dy = (1 + y^2)\, dx.$$

Solution The given differential equation can be written as

$$\frac{dx}{dy} + \frac{x}{1+y^2} = \frac{\tan^{-1}y}{1+y^2} \qquad \text{... (1)}$$