17. Which of the following is a homogeneous differential equation?

(A) $(4x + 6y + 5)\, dy - (3y + 2x + 4)\, dx = 0$

(B) $(xy)\, dx - (x^3 + y^3)\, dy = 0$

(C) $(x^3 + 2y^2)\, dx + 2xy\, dy = 0$

(D) $y^2\, dx + (x^2 - xy - y^2)\, dy = 0$

9.5.3 *Linear differential equations*

A differential equation of the from

$$\frac{dy}{dx} + \mathrm{P}y = \mathrm{Q}$$

where, P and Q are constants or functions of x only, is known as a first order linear differential equation. Some examples of the first order linear differential equation are

$$\frac{dy}{dx} + y = \sin x$$

$$\frac{dy}{dx} + \left(\frac{1}{x}\right)y = e^x$$

$$\frac{dy}{dx} + \left(\frac{y}{x\log x}\right) = \frac{1}{x}$$

Another form of first order linear differential equation is

$$\frac{dx}{dy} + \mathrm{P}_1 x = \mathrm{Q}_1$$

where, P_1 and Q_1 are constants or functions of y only. Some examples of this type of differential equation are

$$\frac{dx}{dy} + x = \cos y$$

$$\frac{dx}{dy} + \frac{-2x}{y} = y^2 e^{-y}$$

To solve the first order linear differential equation of the type

$$\frac{dy}{dx} \quad \mathrm{P}y = \mathrm{Q} \qquad \text{... (1)}$$

Multiply both sides of the equation by a function of x say $g(x)$ to get

$$g(x)\frac{dy}{dx} + \mathrm{P}.(g(x))\, y = \mathrm{Q}\,.\, g(x) \qquad \text{... (2)}$$

Choose $g(x)$ in such a way that R.H.S. becomes a derivative of $y \,.\, g(x)$.

i.e. $$g(x)\frac{dy}{dx} + \mathrm{P}.g(x)\,y = \frac{d}{dx}[y.g(x)]$$

or $$g(x)\frac{dy}{dx} + \mathrm{P}.g(x)\,y = g(x)\frac{dy}{dx} + y\,g'(x)$$

$\Rightarrow$ $$\mathrm{P}.g(x) = g'(x)$$

or $$\mathrm{P} = \frac{g'(x)}{g(x)}$$

Integrating both sides with respect to x, we get

$$\int \mathrm{P}dx = \int \frac{g'(x)}{g(x)}dx$$

or $$\int \mathrm{P}\cdot dx = \log(g(x))$$

or $$\mathrm{g}(x) = e^{\int \mathrm{P}\,dx}$$

On multiplying the equation (1) by $g(x) = e^{\int \mathrm{P}\,dx}$, the L.H.S. becomes the derivative of some function of x and y. This function $g(x) = e^{\int \mathrm{P}\,dx}$ is called *Integrating Factor* (I.F.) of the given differential equation.

Substituting the value of $g(x)$ in equation (2), we get

$$e^{\mathrm{P}\,dx}\frac{dy}{dx} \quad \mathrm{P}e^{\mathrm{P}\,dx}y = \mathrm{Q}\;e^{\mathrm{P}\,dx}$$

or $$\frac{d}{dx}\;y\,e^{\mathrm{P}dx} = \mathrm{Q}e^{\mathrm{P}dx}$$

Integrating both sides with respect to x, we get

$$y\;e^{\mathrm{P}dx} = \quad \mathrm{Q}\,e^{\mathrm{P}dx}\;dx$$

or $$y = e^{\mathrm{P}dx} \quad \mathrm{Q}\,e^{\mathrm{P}dx}\;dx \quad \mathrm{C}$$

which is the general solution of the differential equation.