or $\sin v = \log |x| + \log |C|$

or $\sin v = \log |Cx|$

Replacing v by $\frac{y}{x}$, we get

$$\sin\left(\frac{y}{x}\right) = \log |Cx|$$

which is the general solution of the differential equation (1).

Example 17 Show that the differential equation $2y\, e^{\frac{x}{y}} dx + \left(y - 2x\, e^{\frac{x}{y}}\right) dy = 0$ is homogeneous and find its particular solution, given that, $x = 0$ when $y = 1$.

Solution The given differential equation can be written as

$$\frac{dx}{dy} = \frac{2x\, e^{\frac{x}{y}} - y}{2y\, e^{\frac{x}{y}}} \qquad \dots (1)$$

Let $$F(x, y) = \frac{2xe^{\frac{x}{y}} - y}{2ye^{\frac{x}{y}}}$$

Then $$F(\lambda x, \lambda y) = \frac{\lambda\left(2xe^{\frac{x}{y}} - y\right)}{\lambda\left(2ye^{\frac{x}{y}}\right)} = \lambda^0 [F(x, y)]$$

Thus, F (x, y) is a homogeneous function of degree zero. Therefore, the given differential equation is a homogeneous differential equation.

To solve it, we make the substitution

$$x = vy \qquad \dots (2)$$

Differentiating equation (2) with respect to y, we get

$$\frac{dx}{dy} = v + y\frac{dv}{dy}$$

Substituting the value of x and $\frac{dx}{dy}$ in equation (1), we get

$$v + y\frac{dv}{dy} = \frac{2v\,e^v - 1}{2e^v}$$

or $$y\frac{dv}{dy} = \frac{2v\,e^v - 1}{2e^v} - v$$

or $$y\frac{dv}{dy} = -\frac{1}{2e^v}$$

or $$2e^v\,dv = \frac{-dy}{y}$$

or $$\int 2e^v \cdot dv = -\int \frac{dy}{y}$$

or $$2\,e^v = -\log|y| + C$$

and replacing v by $\frac{x}{y}$, we get

$$2\,e^{\frac{x}{y}} + \log|y| = C \qquad \ldots (3)$$

Substituting $x = 0$ and $y = 1$ in equation (3), we get

$$2\,e^0 + \log|1| = C \Rightarrow C = 2$$

Substituting the value of C in equation (3), we get

$$2\,e^{\frac{x}{y}} + \log|y| = 2$$

which is the particular solution of the given differential equation.

Example 18 Show that the family of curves for which the slope of the tangent at any point (x, y) on it is $\frac{x^2 + y^2}{2xy}$, is given by $x^2 - y^2 = cx$.

Solution We know that the slope of the tangent at any point on a curve is $\frac{dy}{dx}$.

Therefore, $$\frac{dy}{dx} = \frac{x^2 + y^2}{2xy}$$

or $$\frac{dy}{dx} = \frac{1+\frac{y^2}{x^2}}{\frac{2y}{x}} \quad \dots (1)$$

Clearly, (1) is a homogenous differential equation. To solve it we make substitution

$$y = vx$$

Differentiating $y = vx$ with respect to x, we get

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

or $$v + x\frac{dv}{dx} = \frac{1+v^2}{2v}$$

or $$x\frac{dv}{dx} = \frac{1-v^2}{2v}$$

$$\frac{2v}{1-v^2}dv = \frac{dx}{x}$$

or $$\frac{2v}{v^2-1}dv = -\frac{dx}{x}$$

Therefore $$\int\frac{2v}{v^2-1}dv = -\int\frac{1}{x}dx$$

or $$\log|v^2 - 1| = -\log|x| + \log|C_1|$$

or $$\log|(v^2 - 1)(x)| = \log|C_1|$$

or $$(v^2 - 1)\,x = \pm C_1$$

Replacing v by $\frac{y}{x}$, we get

$$\left(\frac{y^2}{x^2}-1\right)x = \pm C_1$$

or $$(y^2 - x^2) = \pm C_1 x \text{ or } x^2 - y^2 = Cx$$