- **20.** In a bank, principal increases continuously at the rate of *r*% per year. Find the value of *r* if Rs 100 double itself in 10 years ($log_e 2 = 0.6931$).
- **21.** In a bank, principal increases continuously at the rate of 5% per year. An amount of Rs 1000 is deposited with this bank, how much will it worth after 10 years $(e^{0.5} = 1.648).$
- **22.** In a culture, the bacteria count is 1,00,000. The number is increased by 10% in 2 hours. In how many hours will the count reach 2,00,000, if the rate of growth of bacteria is proportional to the number present?

23. The general solution of the differential equation $\frac{dy}{dx} = e^{x+y}$ *dx* $=e^{x+y}$ is

 (A) $e^{x} + e^{-y}$ $= C$ (B) $e^x + e^y = C$ (C) $e^{-x} + e^{y}$ $= C$ (D) $e^{-x} + e^{-y} = C$

9.5.2 *Homogeneous differential equations*

Consider the following functions in *x* and *y*

F₁(x, y) = y² + 2xy, F₂(x, y) = 2x - 3y,
F₃(x, y) = cos
$$
\left(\frac{y}{x}\right)
$$
, F₄(x, y) = sin x + cos y

If we replace x and y by λx and λy respectively in the above functions, for any nonzero constant $λ$, we get

value of *P* if its 100 double then in 10 years (log₂ = 0.0951).
\n21. In a bank, principal increases continuous that the rate of 5% per year. An amount of Rs 1000 is deposited with this bank, how much will it worth after 10 years (
$$
e^{0.5}
$$
 = 1.648).
\n22. In a culture, the bacteria count is 1,00,000. The number is increased by 10% in 2 hours. In how many hours will the count reach 2,00,000, if the rate of growth of bacteria is proportional to the number present?
\n23. The general solution of the differential equation $\frac{dy}{dx} = e^{x+y}$ is
\n(A) $e^x + e^y = C$ (B) $e^x + e^y = C$
\n(C) $e^{-x} + e^y = C$ (D) $e^x + e^y = C$
\n9.5.2 Homogeneous differential equations
\nConsider the following functions in *x* and *y*
\nF₁ (*x*, *y*) = *y*² + 2*xy*, F₂ (*x*, *y*) = 2*x* – 3*y*,
\nF₃ (*x*, *y*) = *cos*($\frac{y}{x}$), F₄ (*x*, *y*) = sin *x* + cos *y*
\nIf we replace *x* and *y* by λx and λy respectively in the above functions, for any nonzero constant λ , we get
\nF₁ (*λx*, *λy*) = $\lambda^2 (y^2 + 2xy) = \lambda^2 F_1(x, y)$
\nF₂ (*λx*, *λy*) = *cos*($\frac{\lambda y}{\lambda x}$) = *cos*($\frac{y}{x}$) = *2*⁰ F₃ (*x*, *y*)
\nF₂ (*λx*, *λy*) = *cos*($\frac{\lambda y}{\lambda x}$) = *cos*($\frac{y}{x}$) =

Here, we observe that the functions F_1 , F_2 , F_3 can be written in the form $F(\lambda x, \lambda y) = \lambda^n F(x, y)$ but F_4 can not be written in this form. This leads to the following definition:

A function F(*x*, *y*) is said to be *homogeneous function of degree n* if

 $F(\lambda x, \lambda y) = \lambda^n F(x, y)$ for any nonzero constant λ .

We note that in the above examples, F_1 , F_2 , F_3 are homogeneous functions of degree 2, 1, 0 respectively but F_4 is not a homogeneous function.

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We also observe that

$$
F_1(x, y) = x^2 \left(\frac{y^2}{x^2} + \frac{2y}{x}\right) = x^2 h_1 \left(\frac{y}{x}\right)
$$

or

$$
F_1(x, y) = y^2 \left(1 + \frac{2x}{y}\right) = y^2 h_2 \left(\frac{x}{y}\right)
$$

$$
F_2(x, y) = x^1 \left(2 - \frac{3y}{x}\right) = x^1 h_3 \left(\frac{y}{x}\right)
$$

or

$$
F_2(x, y) = y^1 \left(2\frac{x}{y} - 3\right) = y^1 h_4 \left(\frac{x}{y}\right)
$$

$$
F_3(x, y) = x^0 \cos\left(\frac{y}{x}\right) = x^0 h_5 \left(\frac{y}{x}\right)
$$

For any $n \in \mathbb{N}$
or

$$
F_4(x, y) \neq x^n h_6 \left(\frac{y}{x}\right)
$$
, for any $n \in \mathbb{N}$
Therefore, a function F (x, y) is a homogeneous function of degree n if

$$
F(x, y) = x^n g \left(\frac{y}{x}\right)
$$
 or
$$
y^n h \left(\frac{x}{y}\right)
$$

A differential equation of the form $\frac{dy}{dx} = F(x, y)$ is said to be *homogeneous* if

$$
F(x, y)
$$
 is a homogeneous differential equation of the type

$$
\frac{dy}{dx} = F(x, y) = g \left(\frac{y}{x}\right)
$$
...(1)
We make the substitution

$$
y = y \cdot x
$$

Differentiating equation (2) with respect to x, we get

$$
\frac{dy}{dx} = v + x \frac{dy}{dx}
$$
...(3)
Substituting the value of $\frac{dy}{dx}$ from equation (3) in equation (1), we get

Therefore, a function $F(x, y)$ is a homogeneous function of degree *n* if

$$
F(x, y) = x^{n} g\left(\frac{y}{x}\right) \quad \text{or} \quad y^{n} h\left(\frac{x}{y}\right)
$$

A differential equation of the form $\frac{dy}{dx} = F(x, y)$ is said to be *homogenous* if $F(x, y)$ is a homogenous function of degree zero. or $F_4(x, y) \ne$

Or $F_4(x, y) \ne$

Therefore, a function $F(x, y)$ is a hom
 $F(x, y) =$

A differential equation of the for
 $F(x, y)$ is a homogenous function of do

To solve a homogeneous differential e
 $\frac{dy}{dx} = F(x, y) =$

We mak

To solve a homogeneous differential equation of the type

$$
\frac{dy}{dx} = F(x, y) = g\left(\frac{y}{x}\right) \tag{1}
$$

We make the substitution $y = v \cdot x$... (2) Differentiating equation (2) with respect to *x*, we get

$$
\frac{dy}{dx} = v + x \frac{dv}{dx} \tag{3}
$$

Substituting the value of $\frac{dy}{dx}$ from equation (3) in equation (1), we get

$$
v + x \frac{dv}{dx} = g(v)
$$

$$
x \frac{dv}{dx} = g(v) - v
$$
 ... (4)

or

Separating the variables in equation (4), we get

$$
\frac{dv}{g(v)-v} = \frac{dx}{x}
$$
 ... (5)

Integrating both sides of equation (5), we get

$$
\int \frac{dv}{g(v)-v} = \int \frac{1}{x} dx + C \qquad \qquad \dots (6)
$$

Equation (6) gives general solution (primitive) of the differential equation (1) when we replace v by $\frac{y}{x}$ *x* .

•• Note If the homogeneous differential equation is in the form $\frac{dx}{dy} = F(x, y)$ where, $F(x, y)$ is homogenous function of degree zero, then we make substitution $\frac{x}{-} = v$ *y* $= v$ i.e., $x = vy$ and we proceed further to find the general solution as discussed above by writing $\frac{dx}{dy} = F(x, y) = h\left(\frac{x}{y}\right)$. x

Note If the homogeneous differential

where, F(x, y) is homogeneous functions
 $\frac{x}{y} = v$ i.e., $x = vy$ and we proceed functions

above by writing $\frac{dx}{dy} = F(x, y) = h\left(\frac{dy}{dx}\right)$

Example 15 Show that the differential

and or

or

separating the variables in equation (4), we get

Separating the variables in equation (4), we get
 $\frac{dv}{g(v)-v} = \frac{dx}{x}$ (5)

Integrating both sides of equation (5), we get
 $\int \frac{dv}{g(v)-v} = \int \frac{1}{x} dx + C$ (

Example 15 Show that the differential equation $(x - y) \frac{dy}{dx} = x + 2y$ is homogeneous and solve it.

Solution The given differential equation can be expressed as

$$
\frac{dy}{dx} = \frac{x+2y}{x-y}
$$
 ... (1)
Let

$$
F(x, y) = \frac{x-2y}{x-y}
$$

Now

$$
F(\lambda x, \lambda y) = \frac{(x-2y)}{(x-y)} \qquad 0 \qquad F(x, y)
$$

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Therefore, $F(x, y)$ is a homogenous function of degree zero. So, the given differential equation is a homogenous differential equation.

Alternatively,

$$
\frac{dy}{dx} = \left(\frac{1+\frac{2y}{x}}{1-\frac{y}{x}}\right) = g\left(\frac{y}{x}\right) \quad \dots (2)
$$

R.H.S. of differential equation (2) is of the form $g \frac{y}{x}$ *x* $\frac{y}{x}$ and so it is a homogeneous

function of degree zero. Therefore, equation (1) is a homogeneous differential equation. To solve it we make the substitution

$$
y = vx \tag{3}
$$

Differentiating equation (3) with respect to, *x* we get

$$
\frac{dy}{dx} = v + x \frac{dv}{dx} \tag{4}
$$

Substituting the value of *y* and $\frac{dy}{dx}$ in equation (1) we get

equations is a nonogenous numerical equation.
\nAlternatively,
\n
$$
\frac{dy}{dx} = \left(\frac{1+\frac{2y}{x}}{1-\frac{y}{x}}\right) = g\left(\frac{y}{x}\right) \qquad ... (2)
$$
\nR.H.S. of differential equation (2) is of the form $g\left(\frac{y}{x}\right)$ and so it is a homogeneous function of degree zero. Therefore, equation (1) is a homogeneous differential equation.
\nTo solve it we make the substitution
\n $y = vx$
\nDifferentiating equation (3) with respect to, x we get
\n
$$
\frac{dy}{dx} = v + x\frac{dv}{dx}
$$
...(4)
\nSubstituting the value of y and $\frac{dy}{dx}$ in equation (1) we get
\n
$$
v + x\frac{dv}{dx} = \frac{1+2v}{1-v}
$$
\nor
\n
$$
x\frac{dv}{dx} = \frac{1+2v}{1-v} - v
$$
\nor
\n
$$
x\frac{dv}{dx} = \frac{v^2 - v}{1-v}
$$
\nor
\nIntegrating both sides of equation (5), we get
\n
$$
\frac{v}{v^2 - v} \frac{1}{1} dv = \frac{dx}{x}
$$
\nor
\n
$$
\frac{1}{2} \frac{2v - 1}{v^2 - v} \frac{1}{1} dv = -\log |x| + C_1
$$

or

Integrating both sides of equation (5), we get

$$
\frac{v}{v^2} \frac{1}{v-1} dv = \frac{dx}{x}
$$

or
$$
\frac{1}{2} \frac{2v}{v^2} \frac{1}{v-1} dv = -\log |x| + C_1
$$

or
$$
\frac{1}{2} \frac{2v}{v^2} \frac{1}{v-1} dv \frac{3}{2} \frac{1}{v^2 - v-1} dv \log|x| C_1
$$

or
$$
\frac{1}{2} \log |v^2| v 1 | \frac{3}{2} \frac{1}{\frac{1}{2} \frac{2}{\frac{\sqrt{3}}{2}}} dv \log |x| C_1
$$

or
$$
\frac{1}{2}\log|v^2| v 1| \frac{3}{2} \cdot \frac{2}{\sqrt{3}} \tan^{-1} \frac{2v}{\sqrt{3}} \quad \log|x| C_1
$$

or
$$
\frac{1}{2}\log|v^2| v 1| \frac{1}{2}\log x^2 \sqrt{3} \tan^{-1} \frac{2v}{\sqrt{3}} C_1
$$
 (Why?)

Replacing ν by $\frac{y}{x}$ $\frac{y}{x}$, we get or 2

$$
\int \frac{1}{2} \log \left| \frac{y^2}{x^2} \frac{y}{x} \right| + \frac{1}{2} \log x^2 \sqrt{3} \tan^{-1} \frac{2y}{\sqrt{3}x} + C_1
$$

or

or

or

$$
\log |(y^2 + xy + x^2)| = 2\sqrt{3} \tan^{-1} \left(\frac{2y + x}{\sqrt{3}x}\right) + 2C_1
$$

 $\frac{1}{2} \log \left| \frac{y^2}{x^2} + \frac{y}{x} + 1 \right| x^2 \right| = \sqrt{3} \tan^{-1} \left(\frac{2y + x}{\sqrt{2}} \right) + C_1$

 $\left| \frac{y^2}{2} + \frac{y}{2} + 1 \right| x^2 = \sqrt{3} \tan^{-1} \left(\frac{2y + x}{\sqrt{3}} \right)$ $\left(\frac{y^2}{x^2} + \frac{y}{x} + 1\right) x^2 = \sqrt{3} \tan^{-1} \left(\frac{2y + x}{\sqrt{3}x}\right) +$

$$
\log |(x^2 + xy + y^2)| = 2\sqrt{3} \tan^{-1} \left(\frac{x + 2y}{\sqrt{3}x} \right) + C
$$

 $2\frac{8}{x^2}$ x) $\frac{1}{x^3}$

which is the general solution of the differential equation (1)

Example 16 Show that the differential equation $x \cos\left(\frac{y}{x}\right) \frac{dy}{dx} = y \cos\left(\frac{y}{x}\right) + x$ is homogeneous and solve it. or $\frac{1}{2} \log \left| \frac{y^2}{x^2} \right| \frac{y}{x} = 1 \right| = \frac{1}{2}$

or $\log |(y^2 + xy + x^2)| = 2$

or $\log |(x^2 + xy + y^2)| = 2$

which is the general solution of the differential equation
 Example 16 Show that the differential equation
 Solution or $\frac{1}{2} \log |y^2 + y + 1|^2 \frac{2}{3} \frac{1}{\sqrt{3}} \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \log |x| + C_1$

or $\frac{1}{2} \log |y^2 + y + 1| \frac{3}{2} \frac{1}{\sqrt{3}} \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \log |x| + C_1$

or $\frac{1}{2} \log |y^2 + y + 1| \frac{1}{2} \log x^2 + \sqrt{3} \tan^{-1} \frac{2y -$

Solution The given differential equation can be written as

$$
\frac{dy}{dx} = \frac{y \cos\left(\frac{y}{x}\right) + x}{x \cos\left(\frac{y}{x}\right)} \quad \dots (1)
$$

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It is a differential equation of the form $\frac{dy}{dx} = F(x, y)$.

$$
(x, y) = \frac{y \cos\left(\frac{y}{x}\right) + x}{x \cos\left(\frac{y}{x}\right)}
$$

Here **F**(

Replacing *x* by λx and *y* by λy , we get

$$
F(\lambda x, \lambda y) = \frac{\lambda [y \cos \left(\frac{y}{x}\right) + x]}{\lambda \left(x \cos \frac{y}{x}\right)} = \lambda^{0} [F(x, y)]
$$

Thus, $F(x, y)$ is a homogeneous function of degree zero.

Therefore, the given differential equation is a homogeneous differential equation. To solve it we make the substitution

$$
y = vx \tag{2}
$$

Differentiating equation (2) with respect to *x*, we get

$$
\frac{dy}{dx} = v + x \frac{dv}{dx} \tag{3}
$$

Substituting the value of *y* and $\frac{dy}{dx}$ in equation (1), we get

Here
$$
F(x, y) = \frac{y \cos(\frac{y}{x}) + x}{x \cos(\frac{y}{x})}
$$

\nHere $F(x, y) = \frac{y \cos(\frac{y}{x}) + x}{x \cos(\frac{y}{x})}$
\nReplacing x by λx and y by λy , we get
\n $F(\lambda x, \lambda y) = \frac{\lambda[y \cos(\frac{y}{x}) + x]}{\lambda[x \cos \frac{y}{x}]} = \lambda^0 (F(x, y))$
\nThus, $F(x, y)$ is a homogeneous function of degree zero.
\nTherefore, the given differential equation is a homogeneous differential equation.
\nTo solve it we make the substitution
\n $y = vx$...(2)
\nDifferentiating equation (2) with respect to x, we get
\n $\frac{dy}{dx} = v + x \frac{dv}{dx}$...(3)
\nSubstituting the value of y and $\frac{dy}{dx}$ in equation (1), we get
\n $v + x \frac{dv}{dx} = \frac{v \cos v + 1}{\cos v}$
\nor
\n $x \frac{dv}{dx} = \frac{v \cos v + 1}{\cos v} - v$
\nor
\n $x \frac{dv}{dx} = \frac{1}{\cos v}$
\nor
\n $\cos v dv = \int \frac{1}{x} dx$

or

or

Therefore