20. In a bank, principal increases continuously at the rate of r% per year. Find the value of r if Rs 100 double itself in 10 years ($\log_e 2 = 0.6931$).

21. In a bank, principal increases continuously at the rate of 5% per year. An amount of Rs 1000 is deposited with this bank, how much will it worth after 10 years ($e^{0.5} = 1.648$).

22. In a culture, the bacteria count is 1,00,000. The number is increased by 10% in 2 hours. In how many hours will the count reach 2,00,000, if the rate of growth of bacteria is proportional to the number present?

23. The general solution of the differential equation $\frac{dy}{dx} = e^{x+y}$ is

(A) $e^x + e^{-y} = C$ (B) $e^x + e^y = C$

(C) $e^{-x} + e^y = C$ (D) $e^{-x} + e^{-y} = C$

9.5.2 *Homogeneous differential equations*

Consider the following functions in x and y

$$F_1(x, y) = y^2 + 2xy, \qquad F_2(x, y) = 2x - 3y,$$

$$F_3(x, y) = \cos\left(\frac{y}{x}\right), \qquad F_4(x, y) = \sin x + \cos y$$

If we replace x and y by λx and λy respectively in the above functions, for any nonzero constant λ, we get

$$F_1(\lambda x, \lambda y) = \lambda^2 (y^2 + 2xy) = \lambda^2 F_1(x, y)$$

$$F_2(\lambda x, \lambda y) = \lambda (2x - 3y) = \lambda F_2(x, y)$$

$$F_3(\lambda x, \lambda y) = \cos\left(\frac{\lambda y}{\lambda x}\right) = \cos\left(\frac{y}{x}\right) = \lambda^0 F_3(x, y)$$

$$F_4(\lambda x, \lambda y) = \sin \lambda x + \cos \lambda y \neq \lambda^n F_4(x, y), \text{ for any } n \in \mathbf{N}$$

Here, we observe that the functions F_1, F_2, F_3 can be written in the form $F(\lambda x, \lambda y) = \lambda^n F(x, y)$ but F_4 can not be written in this form. This leads to the following definition:

A function $F(x, y)$ is said to be *homogeneous function of degree n* if

$F(\lambda x, \lambda y) = \lambda^n F(x, y)$ for any nonzero constant λ.

We note that in the above examples, F_1, F_2, F_3 are homogeneous functions of degree 2, 1, 0 respectively but F_4 is not a homogeneous function.

We also observe that

$$F_1(x, y) = x^2\left(\frac{y^2}{x^2}+\frac{2y}{x}\right)=x^2h_1\left(\frac{y}{x}\right)$$

or $$F_1(x, y) = y^2\left(1+\frac{2x}{y}\right)=y^2h_2\left(\frac{x}{y}\right)$$

$$F_2(x, y) = x^1\left(2-\frac{3y}{x}\right)=x^1h_3\left(\frac{y}{x}\right)$$

or $$F_2(x, y) = y^1\left(2\frac{x}{y}-3\right)=y^1h_4\left(\frac{x}{y}\right)$$

$$F_3(x, y) = x^0\cos\left(\frac{y}{x}\right)=x^0\ h_5\left(\frac{y}{x}\right)$$

$$F_4(x, y) \neq x^n\ h_6\left(\frac{y}{x}\right),\ \text{for any } n \in \mathbf{N}$$

or $$F_4(x, y) \neq y^n\ h_7\left(\frac{x}{y}\right),\ \text{for any } n \in \mathbf{N}$$

Therefore, a function F (x, y) is a homogeneous function of degree n if

$$F(x, y) = x^n g\left(\frac{y}{x}\right) \quad \text{or} \quad y^n h\left(\frac{x}{y}\right)$$

A differential equation of the form $\frac{dy}{dx}$ = F (x, y) is said to be *homogenous* if F(x, y) is a homogenous function of degree zero.

To solve a homogeneous differential equation of the type

$$\frac{dy}{dx}=F(x,y) = g\left(\frac{y}{x}\right) \qquad \text{... (1)}$$

We make the substitution $y = v\,.\,x$ ... (2)

Differentiating equation (2) with respect to x, we get

$$\frac{dy}{dx} = v+x\frac{dv}{dx} \qquad \text{... (3)}$$

Substituting the value of $\frac{dy}{dx}$ from equation (3) in equation (1), we get

$$v + x\frac{dv}{dx} = g(v)$$

or $$x\frac{dv}{dx} = g(v) - v \qquad \text{... (4)}$$

Separating the variables in equation (4), we get

$$\frac{dv}{g(v) - v} = \frac{dx}{x} \qquad \text{... (5)}$$

Integrating both sides of equation (5), we get

$$\int \frac{dv}{g(v) - v} = \int \frac{1}{x}\,dx + \text{C} \qquad \text{... (6)}$$

Equation (6) gives general solution (primitive) of the differential equation (1) when we replace v by $\frac{y}{x}$.

Note If the homogeneous differential equation is in the form $\frac{dx}{dy} = \text{F}(x, y)$ where, F (x, y) is homogenous function of degree zero, then we make substitution $\frac{x}{y} = v$ i.e., $x = vy$ and we proceed further to find the general solution as discussed above by writing $\frac{dx}{dy} = \text{F}(x, y) = h\left(\frac{x}{y}\right)$.

Example 15 Show that the differential equation $(x - y)\frac{dy}{dx} = x + 2y$ is homogeneous and solve it.

Solution The given differential equation can be expressed as

$$\frac{dy}{dx} = \frac{x + 2y}{x - y} \qquad \text{... (1)}$$

Let $$\text{F}(x, y) = \frac{x \quad 2y}{x \quad y}$$

Now $$\text{F}(\lambda x, \lambda y) = \frac{(x \quad 2y)}{(x \quad y)} \quad {}^{0}\, F(x, y)$$

Therefore, F (x, y) is a homogenous function of degree zero. So, the given differential equation is a homogenous differential equation.

Alternatively,

$$\frac{dy}{dx} = \left(\frac{1+\frac{2y}{x}}{1-\frac{y}{x}}\right) = g\left(\frac{y}{x}\right) \qquad \text{... (2)}$$

R.H.S. of differential equation (2) is of the form $g \;\; \frac{y}{x}$ and so it is a homogeneous function of degree zero. Therefore, equation (1) is a homogeneous differential equation. To solve it we make the substitution

$$y = vx \qquad \text{... (3)}$$

Differentiating equation (3) with respect to, x we get

$$\frac{dy}{dx} = v + x\frac{dv}{dx} \qquad \text{... (4)}$$

Substituting the value of y and $\frac{dy}{dx}$ in equation (1) we get

$$v + x\frac{dv}{dx} = \frac{1+2v}{1-v}$$

or $$x\frac{dv}{dx} = \frac{1+2v}{1-v} - v$$

or $$x\frac{dv}{dx} = \frac{v^2 \quad v \quad 1}{1 \quad v}$$

or $$\frac{v \quad 1}{v^2 \quad v \quad 1}dv = \frac{dx}{x}$$

Integrating both sides of equation (5), we get

$$\frac{v \quad 1}{v^2 \quad v \quad 1}dv = \qquad \frac{dx}{x}$$

or $$\frac{1}{2}\;\frac{2v \quad 1 \quad 3}{v^2 \quad v \quad 1}dv = -\log|x| + C_1$$

or $$\frac{1}{2}\ \frac{2v\quad 1}{v^2\quad v\quad 1}dv\quad \frac{3}{2}\ \frac{1}{v^2\quad v\quad 1}dv\qquad \log|x|\quad C_1$$

or $$\frac{1}{2}\log|v^2\quad v\quad 1|\quad \frac{3}{2}\ \frac{1}{v\quad \frac{1}{2}^{\ 2}\qquad \frac{\sqrt{3}}{2}^{\ 2}}dv\qquad \log|x|\quad C_1$$

or $$\frac{1}{2}\log|v^2\quad v\quad 1|\quad \frac{3}{2}.\frac{2}{\sqrt{3}}\tan^{\ 1}\frac{2v\quad 1}{\sqrt{3}}\qquad \log|x|\quad C_1$$

or $$\frac{1}{2}\log|v^2\quad v\quad 1|\quad \frac{1}{2}\log x^2\quad \sqrt{3}\tan^{\ 1}\frac{2v\quad 1}{\sqrt{3}}\quad C_1 \qquad \text{(Why?)}$$

Replacing v by $\frac{y}{x}$, we get

or $$\frac{1}{2}\log\left|\frac{y^2}{x^2}\quad \frac{y}{x}\quad 1\right|\quad \frac{1}{2}\log x^2\quad \sqrt{3}\tan^{\ 1}\frac{2y\quad x}{\sqrt{3}x}\quad C_1$$

or $$\frac{1}{2}\log\left|\left(\frac{y^2}{x^2}+\frac{y}{x}+1\right)x^2\right| = \sqrt{3}\tan^{-1}\left(\frac{2y+x}{\sqrt{3}x}\right)+C_1$$

or $$\log|(y^2+xy+x^2)| = 2\sqrt{3}\tan^{-1}\left(\frac{2y+x}{\sqrt{3}x}\right)+2C_1$$

or $$\log|(x^2+xy+y^2)| = 2\sqrt{3}\tan^{-1}\left(\frac{x+2y}{\sqrt{3}x}\right)+C$$

which is the general solution of the differential equation (1)

Example 16 Show that the differential equation $x\cos\left(\frac{y}{x}\right)\frac{dy}{dx} = y\cos\left(\frac{y}{x}\right)+x$ is homogeneous and solve it.

Solution The given differential equation can be written as

$$\frac{dy}{dx} = \frac{y\cos\left(\frac{y}{x}\right)+x}{x\cos\left(\frac{y}{x}\right)} \qquad \text{... (1)}$$

It is a differential equation of the form $\frac{dy}{dx} = F(x, y)$.

Here $$F(x, y) = \frac{y\cos\left(\frac{y}{x}\right) + x}{x\cos\left(\frac{y}{x}\right)}$$

Replacing x by λx and y by λy, we get

$$F(\lambda x, \lambda y) = \frac{\lambda\left[y\cos\left(\frac{y}{x}\right) + x\right]}{\lambda\left(x\cos\frac{y}{x}\right)} = \lambda^0[F(x, y)]$$

Thus, $F(x, y)$ is a homogeneous function of degree zero.

Therefore, the given differential equation is a homogeneous differential equation. To solve it we make the substitution

$$y = vx \qquad \text{... (2)}$$

Differentiating equation (2) with respect to x, we get

$$\frac{dy}{dx} = v + x\frac{dv}{dx} \qquad \text{... (3)}$$

Substituting the value of y and $\frac{dy}{dx}$ in equation (1), we get

$$v + x\frac{dv}{dx} = \frac{v\cos v + 1}{\cos v}$$

or $$x\frac{dv}{dx} = \frac{v\cos v + 1}{\cos v} - v$$

or $$x\frac{dv}{dx} = \frac{1}{\cos v}$$

or $$\cos v\ dv = \frac{dx}{x}$$

Therefore $$\int \cos v\ dv = \int \frac{1}{x}\, dx$$