Example 10 Find the general solution of the differential equation $\frac{dy}{dx} = \frac{1+y^2}{1+x^2}$.

Solution Since $1 + y^2 \neq 0$, therefore separating the variables, the given differential equation can be written as

$$\frac{dy}{1+y^2} = \frac{dx}{1+x^2} \qquad \text{... (1)}$$

Integrating both sides of equation (1), we get

$$\int \frac{dy}{1+y^2} = \int \frac{dx}{1+x^2}$$

or $$\tan^{-1} y = \tan^{-1} x + C$$

which is the general solution of equation (1).

Example 11 Find the particular solution of the differential equation $\frac{dy}{dx} = -4xy^2$ given that $y = 1$, when $x = 0$.

Solution If $y \neq 0$, the given differential equation can be written as

$$\frac{dy}{y^2} = -4x\,dx \qquad \text{... (1)}$$

Integrating both sides of equation (1), we get

$$\int \frac{dy}{y^2} = -4\int x\,dx$$

or $$-\frac{1}{y} = -2x^2 + C$$

or $$y = \frac{1}{2x^2 - C} \qquad \text{... (2)}$$

Substituting $y = 1$ and $x = 0$ in equation (2), we get, $C = -1$.

Now substituting the value of C in equation (2), we get the particular solution of the given differential equation as $y = \frac{1}{2x^2+1}$.

Example 12 Find the equation of the curve passing through the point (1, 1) whose differential equation is $x\,dy = (2x^2 + 1)\,dx$ $(x \neq 0)$.

Solution The given differential equation can be expressed as

$$dy^* = \frac{2x^2 \quad 1}{x}\, dx^*$$

or $$dy = \left(2x + \frac{1}{x}\right) dx \qquad \text{... (1)}$$

Integrating both sides of equation (1), we get

$$\int dy = \int \left(2x + \frac{1}{x}\right) dx$$

or $$y = x^2 + \log|x| + C \qquad \text{... (2)}$$

Equation (2) represents the family of solution curves of the given differential equation but we are interested in finding the equation of a particular member of the family which passes through the point (1, 1). Therefore substituting $x = 1$, $y = 1$ in equation (2), we get C = 0.

Now substituting the value of C in equation (2) we get the equation of the required curve as $y = x^2 + \log|x|$.

Example 13 Find the equation of a curve passing through the point (–2, 3), given that the slope of the tangent to the curve at any point (x, y) is $\frac{2x}{y^2}$.

Solution We know that the slope of the tangent to a curve is given by $\frac{dy}{dx}$.

so, $$\frac{dy}{dx} = \frac{2x}{y^2} \qquad \text{... (1)}$$

Separating the variables, equation (1) can be written as

$$y^2\, dy = 2x\, dx \qquad \text{... (2)}$$

Integrating both sides of equation (2), we get

$$\int y^2 dy = \int 2x\, dx$$

or $$\frac{y^3}{3} = x^2 + C \qquad \text{... (3)}$$

* The notation $\frac{dy}{dx}$ due to Leibnitz is extremely flexible and useful in many calculation and formal transformations, where, we can deal with symbols *dy* and *dx* exactly as if they were ordinary numbers. By treating *dx* and *dy* like separate entities, we can give neater expressions to many calculations.

Refer: Introduction to Calculus and Analysis, volume-I page 172, By Richard Courant, Fritz John Spinger – Verlog New York.

Substituting $x = -2$, $y = 3$ in equation (3), we get C = 5.

Substituting the value of C in equation (3), we get the equation of the required curve as

$$\frac{y^3}{3} = x^2 + 5 \quad \text{or} \quad y = (3x^2 + 15)^{\frac{1}{3}}$$

Example 14 In a bank, principal increases continuously at the rate of 5% per year. In how many years Rs 1000 double itself?

Solution Let P be the principal at any time t. According to the given problem,

$$\frac{dp}{dt} = \left(\frac{5}{100}\right) \times \mathrm{P}$$

or $$\frac{dp}{dt} = \frac{\mathrm{P}}{20} \qquad \ldots (1)$$

separating the variables in equation (1), we get

$$\frac{dp}{\mathrm{P}} = \frac{dt}{20} \qquad \ldots (2)$$

Integrating both sides of equation (2), we get

$$\log \mathrm{P} = \frac{t}{20} + \mathrm{C}_1$$

or $$\mathrm{P} = e^{\frac{t}{20}} \cdot e^{\mathrm{C}_1}$$

or $$\mathrm{P} = \mathrm{C}\, e^{\frac{t}{20}} \quad (\text{where } e^{\mathrm{C}_1} = \mathrm{C}) \qquad \ldots (3)$$

Now $\mathrm{P} = 1000$, when $t = 0$

Substituting the values of P and t in (3), we get C = 1000. Therefore, equation (3), gives

$$\mathrm{P} = 1000\, e^{\frac{t}{20}}$$

Let t years be the time required to double the principal. Then

$$2000 = 1000\, e^{\frac{t}{20}} \Rightarrow t = 20 \log_e 2$$

EXERCISE 9.4

For each of the differential equations in Exercises 1 to 10, find the general solution:

1. $\dfrac{dy}{dx} = \dfrac{1 - \cos x}{1 + \cos x}$

2. $\dfrac{dy}{dx} = \sqrt{4 - y^2} \quad (-2 < y < 2)$