4.6.1 *Adjoint of a matrix*

Definition 3 The adjoint of a square matrix $A = [a_{ij}]_{n \times n}$ is defined as the transpose of the matrix $[A_{ij}]_{n \times n}$, where A_{ij} is the cofactor of the element a_{ij}. Adjoint of the matrix A is denoted by *adj* A.

Let
$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Then
$$adj\ A = \text{Transpose of} \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

Example 23 Find *adj* A for $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$

Solution We have $A_{11} = 4, A_{12} = -1, A_{21} = -3, A_{22} = 2$

Hence
$$adj\ A = \begin{bmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ -1 & 2 \end{bmatrix}$$

Remark For a square matrix of order 2, given by

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

The *adj* A can also be obtained by interchanging a_{11} and a_{22} and by changing signs of a_{12} and a_{21}, i.e.,

$$adj\ A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

Change sign Interchange

We state the following theorem without proof.

Theorem 1 If A be any given square matrix of order n, then

$$A(adj\ A) = (adj\ A)\ A = |A|\,I,$$

where I is the identity matrix of order n

Verification

Let $\quad A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then *adj* $A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$

Since sum of product of elements of a row (or a column) with corresponding cofactors is equal to $|A|$ and otherwise zero, we have

$$A\,(adj\ A) = \begin{bmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{bmatrix} = |A| \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = |A|\ I$$

Similarly, we can show $(adj\ A)\,A = |A|\ I$

Hence $A\,(adj\ A) = (adj\ A)\,A = |A|\ I$

Definition 4 A square matrix A is said to be singular if $|A| = 0$.

For example, the determinant of matrix $A = \begin{bmatrix} 1 & 2 \\ 4 & 8 \end{bmatrix}$ is zero

Hence A is a singular matrix.

Definition 5 A square matrix A is said to be non-singular if $|A| \neq 0$

Let $\quad A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. Then $|A| = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4 - 6 = -2 \neq 0$.

Hence A is a nonsingular matrix

We state the following theorems without proof.

Theorem 2 If A and B are nonsingular matrices of the same order, then AB and BA are also nonsingular matrices of the same order.

Theorem 3 The determinant of the product of matrices is equal to product of their respective determinants, that is, $|AB| = |A|\ |B|$, where A and B are square matrices of the same order

Remark We know that $(adj\ A)\,A = |A|\ I = \begin{bmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{bmatrix}, |A| \neq 0$

Writing determinants of matrices on both sides, we have

$$|(adj\ \mathrm{A})\,\mathrm{A}| = \begin{vmatrix} |\mathrm{A}| & 0 & 0 \\ 0 & |\mathrm{A}| & 0 \\ 0 & 0 & |\mathrm{A}| \end{vmatrix}$$

i.e. $$|(adj\ \mathrm{A})|\ |\mathrm{A}| = |\mathrm{A}|^3 \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \qquad \text{(Why?)}$$

i.e. $$|(adj\ \mathrm{A})|\ |\mathrm{A}| = |\mathrm{A}|^3\ (1)$$

i.e. $$|(adj\ \mathrm{A})| = |\mathrm{A}|^2$$

In general, if A is *a* square matrix of order n, then $|adj(\mathrm{A})| = |\mathrm{A}|^{n-1}$.

Theorem 4 A square matrix A is invertible if and only if A is nonsingular matrix.

Proof Let A be invertible matrix of order n and I be the identity matrix of order n. Then, there exists a square matrix B of order n such that AB = BA = I

Now $\quad$ AB = I. So $|\mathrm{AB}| = |\mathrm{I}|$ or $|\mathrm{A}|\ |\mathrm{B}| = 1$ $\quad$ (since $|\mathrm{I}|=1, |\mathrm{AB}|=|\mathrm{A}||\mathrm{B}|$)

This gives $\quad |\mathrm{A}| \neq 0$. Hence A is nonsingular.

Conversely, let A be nonsingular. Then $|\mathrm{A}| \neq 0$

Now $\quad \mathrm{A}\,(adj\ \mathrm{A}) = (adj\ \mathrm{A})\,\mathrm{A} = |\mathrm{A}|\,\mathrm{I}$ $\quad$ (Theorem 1)

or $$\mathrm{A}\left(\frac{1}{|\mathrm{A}|}adj\ \mathrm{A}\right) = \left(\frac{1}{|\mathrm{A}|}adj\ \mathrm{A}\right)\mathrm{A} = \mathrm{I}$$

or $$\mathrm{AB} = \mathrm{BA} = \mathrm{I}, \text{ where } \mathrm{B} = \frac{1}{|\mathrm{A}|}adj\ \mathrm{A}$$

Thus $$\text{A is invertible and } \mathrm{A}^{-1} = \frac{1}{|\mathrm{A}|}adj\ \mathrm{A}$$

Example 24 If $\mathrm{A} = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$, then verify that A *adj* A = $|\mathrm{A}|$ I. Also find A^{-1}.

Solution We have $|\mathrm{A}| = 1\,(16-9) - 3\,(4-3) + 3\,(3-4) = 1 \neq 0$