(iii) $\begin{vmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ -2 & 3 & 0 \end{vmatrix}$ (iv) $\begin{vmatrix} 2 & -1 & -2 \\ 0 & 2 & -1 \\ 3 & -5 & 0 \end{vmatrix}$

6. If $A = \begin{bmatrix} 1 & 1 & -2 \\ 2 & 1 & -3 \\ 5 & 4 & -9 \end{bmatrix}$, find $|A|$

7. Find values of x, if

(i) $\begin{vmatrix} 2 & 4 \\ 5 & 1 \end{vmatrix} = \begin{vmatrix} 2x & 4 \\ 6 & x \end{vmatrix}$ (ii) $\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = \begin{vmatrix} x & 3 \\ 2x & 5 \end{vmatrix}$

8. If $\begin{vmatrix} x & 2 \\ 18 & x \end{vmatrix} = \begin{vmatrix} 6 & 2 \\ 18 & 6 \end{vmatrix}$, then x is equal to

(A) 6 (B) ± 6 (C) -6 (D) 0

4.3 Properties of Determinants

In the previous section, we have learnt how to expand the determinants. In this section, we will study some properties of determinants which simplifies its evaluation by obtaining maximum number of zeros in a row or a column. These properties are true for determinants of any order. However, we shall restrict ourselves upto determinants of order 3 only.

Property 1 The value of the determinant remains unchanged if its rows and columns are interchanged.

Verification Let $\Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

Expanding along first row, we get

$$\Delta = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$= a_1 (b_2 c_3 - b_3 c_2) - a_2 (b_1 c_3 - b_3 c_1) + a_3 (b_1 c_2 - b_2 c_1)$$

By interchanging the rows and columns of Δ, we get the determinant

$$\Delta_1 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Expanding Δ_1 along first column, we get

$$\Delta_1 = a_1\,(b_2\,c_3 - c_2\,b_3) - a_2\,(b_1\,c_3 - b_3\,c_1) + a_3\,(b_1\,c_2 - b_2\,c_1)$$

Hence $\Delta = \Delta_1$

Remark It follows from above property that if A is a square matrix, then det (A) = det (A′), where A′ = transpose of A.

Note If R_i = ith row and C_i = ith column, then for interchange of row and columns, we will symbolically write $C_i \leftrightarrow R_i$

Let us verify the above property by example.

Example 6 Verify Property 1 for $\Delta = \begin{vmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{vmatrix}$

Solution Expanding the determinant along first row, we have

$$\Delta = 2\begin{vmatrix} 0 & 4 \\ 5 & -7 \end{vmatrix} - (-3)\begin{vmatrix} 6 & 4 \\ 1 & -7 \end{vmatrix} + 5\begin{vmatrix} 6 & 0 \\ 1 & 5 \end{vmatrix}$$

$$= 2\,(0 - 20) + 3\,(-42 - 4) + 5\,(30 - 0)$$

$$= -40 - 138 + 150 = -28$$

By interchanging rows and columns, we get

$$\Delta_1 = \begin{vmatrix} 2 & 6 & 1 \\ -3 & 0 & 5 \\ 5 & 4 & -7 \end{vmatrix} \quad \text{(Expanding along first column)}$$

$$= 2\begin{vmatrix} 0 & 5 \\ 4 & -7 \end{vmatrix} - (-3)\begin{vmatrix} 6 & 1 \\ 4 & -7 \end{vmatrix} + 5\begin{vmatrix} 6 & 1 \\ 0 & 5 \end{vmatrix}$$

$$= 2\,(0 - 20) + 3\,(-42 - 4) + 5\,(30 - 0)$$

$$= -40 - 138 + 150 = -28$$

Clearly $\Delta = \Delta_1$

Hence, Property 1 is verified.

Property 2 If any two rows (or columns) of a determinant are interchanged, then sign of determinant changes.

Verification Let $\Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

Expanding along first row, we get

$$\Delta = a_1 (b_2 c_3 - b_3 c_2) - a_2 (b_1 c_3 - b_3 c_1) + a_3 (b_1 c_2 - b_2 c_1)$$

Interchanging first and third rows, the new determinant obtained is given by

$$\Delta_1 = \begin{vmatrix} c_1 & c_2 & c_3 \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix}$$

Expanding along third row, we get

$$\Delta_1 = a_1 (c_2 b_3 - b_2 c_3) - a_2 (c_1 b_3 - c_3 b_1) + a_3 (b_2 c_1 - b_1 c_2)$$
$$= - [a_1 (b_2 c_3 - b_3 c_2) - a_2 (b_1 c_3 - b_3 c_1) + a_3 (b_1 c_2 - b_2 c_1)]$$

Clearly $\Delta_1 = - \Delta$

Similarly, we can verify the result by interchanging any two columns.

Note We can denote the interchange of rows by $R_i \leftrightarrow R_j$ and interchange of columns by $C_i \leftrightarrow C_j$.

Example 7 Verify Property 2 for $\Delta = \begin{vmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{vmatrix}$.

Solution $\Delta = \begin{vmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{vmatrix} = -28$ (See Example 6)

Interchanging rows R_2 and R_3 i.e., $R_2 \leftrightarrow R_3$, we have

$$\Delta_1 = \begin{vmatrix} 2 & -3 & 5 \\ 1 & 5 & -7 \\ 6 & 0 & 4 \end{vmatrix}$$

Expanding the determinant Δ_1 along first row, we have

$$\Delta_1 = 2\begin{vmatrix} 5 & -7 \\ 0 & 4 \end{vmatrix} - (-3)\begin{vmatrix} 1 & -7 \\ 6 & 4 \end{vmatrix} + 5\begin{vmatrix} 1 & 5 \\ 6 & 0 \end{vmatrix}$$
$$= 2\,(20 - 0) + 3\,(4 + 42) + 5\,(0 - 30)$$
$$= 40 + 138 - 150 = 28$$

Clearly $$\Delta_1 = -\Delta$$

Hence, Property 2 is verified.

Property 3 If any two rows (or columns) of a determinant are identical (all corresponding elements are same), then value of determinant is zero.

Proof If we interchange the identical rows (or columns) of the determinant Δ, then Δ does not change. However, by Property 2, it follows that Δ has changed its sign

Therefore $$\Delta = -\Delta$$

or $$\Delta = 0$$

Let us verify the above property by an example.

Example 8 Evaluate $\Delta = \begin{vmatrix} 3 & 2 & 3 \\ 2 & 2 & 3 \\ 3 & 2 & 3 \end{vmatrix}$

Solution Expanding along first row, we get

$$\Delta = 3\,(6 - 6) - 2\,(6 - 9) + 3\,(4 - 6)$$
$$= 0 - 2\,(-3) + 3\,(-2) = 6 - 6 = 0$$

Here R_1 and R_3 are identical.

Property 4 If each element of a row (or a column) of a determinant is multiplied by a constant k, then its value gets multiplied by k.

Verification Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

and Δ_1 be the determinant obtained by multiplying the elements of the first row by k. Then

$$\Delta_1 = \begin{vmatrix} k\,a_1 & k\,b_1 & k\,c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Expanding along first row, we get

$$\Delta_1 = k\,a_1\,(b_2\,c_3 - b_3\,c_2) - k\,b_1\,(a_2\,c_3 - c_2\,a_3) + k\,c_1\,(a_2\,b_3 - b_2\,a_3)$$
$$= k\,[a_1\,(b_2\,c_3 - b_3\,c_2) - b_1\,(a_2\,c_3 - c_2\,a_3) + c_1\,(a_2\,b_3 - b_2\,a_3)]$$
$$= k\,\Delta$$

Hence
$$\begin{vmatrix} k\,a_1 & k\,b_1 & k\,c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = k \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Remarks

(i) By this property, we can take out any common factor from any one row or any one column of a given determinant.

(ii) If corresponding elements of any two rows (or columns) of a determinant are proportional (in the same ratio), then its value is zero. For example

$$\Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ k\,a_1 & k\,a_2 & k\,a_3 \end{vmatrix} = 0 \text{ (rows } R_1 \text{ and } R_2 \text{ are proportional)}$$

Example 9 Evaluate $\begin{vmatrix} 102 & 18 & 36 \\ 1 & 3 & 4 \\ 17 & 3 & 6 \end{vmatrix}$

Solution Note that $\begin{vmatrix} 102 & 18 & 36 \\ 1 & 3 & 4 \\ 17 & 3 & 6 \end{vmatrix} = \begin{vmatrix} 6(17) & 6(3) & 6(6) \\ 1 & 3 & 4 \\ 17 & 3 & 6 \end{vmatrix} = 6\begin{vmatrix} 17 & 3 & 6 \\ 1 & 3 & 4 \\ 17 & 3 & 6 \end{vmatrix} = 0$

(Using Properties 3 and 4)

Property 5 If some or all elements of a row or column of a determinant are expressed as sum of two (or more) terms, then the determinant can be expressed as sum of two (or more) determinants.

For example, $\begin{vmatrix} a_1+\lambda_1 & a_2+\lambda_2 & a_3+\lambda_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

Verification L.H.S. = $\begin{vmatrix} a_1+\lambda_1 & a_2+\lambda_2 & a_3+\lambda_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

Expanding the determinants along the first row, we get

$$\Delta = (a_1 + \lambda_1)(b_2 c_3 - c_2 b_3) - (a_2 + \lambda_2)(b_1 c_3 - b_3 c_1) + (a_3 + \lambda_3)(b_1 c_2 - b_2 c_1)$$

$$= a_1 (b_2 c_3 - c_2 b_3) - a_2 (b_1 c_3 - b_3 c_1) + a_3 (b_1 c_2 - b_2 c_1) + \lambda_1 (b_2 c_3 - c_2 b_3) - \lambda_2 (b_1 c_3 - b_3 c_1) + \lambda_3 (b_1 c_2 - b_2 c_1)$$

(by rearranging terms)

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \text{R.H.S.}$$

Similarly, we may verify Property 5 for other rows or columns.

Example 10 Show that $\begin{vmatrix} a & b & c \\ a+2x & b+2y & c+2z \\ x & y & z \end{vmatrix} = 0$

Solution We have $\begin{vmatrix} a & b & c \\ a+2x & b+2y & c+2z \\ x & y & z \end{vmatrix} = \begin{vmatrix} a & b & c \\ a & b & c \\ x & y & z \end{vmatrix} + \begin{vmatrix} a & b & c \\ 2x & 2y & 2z \\ x & y & z \end{vmatrix}$

(by Property 5)

$$= 0 + 0 = 0$$

(Using Property 3 and Property 4)

Property 6 If, to each element of any row or column of a determinant, the equimultiples of corresponding elements of other row (or column) are added, then value of determinant remains the same, i.e., the value of determinant remain same if we apply the operation $R_i \to R_i + kR_j$ or $C_i \to C_i + k\,C_j$.

Verification

Let $\Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ and $\Delta_1 = \begin{vmatrix} a_1 + k c_1 & a_2 + k c_2 & a_3 + k c_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$,

where Δ_1 is obtained by the operation $R_1 \to R_1 + kR_3$.

Here, we have multiplied the elements of the third row (R_3) by a constant k and added them to the corresponding elements of the first row (R_1).

Symbolically, we write this operation as $R_1 \to R_1 + k\,R_3$.

Now, again

$$\Delta_1 = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} k\,c_1 & k\,c_2 & k\,c_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \quad \text{(Using Property 5)}$$

$$= \Delta + 0 \qquad \text{(since } R_1 \text{ and } R_3 \text{ are proportional)}$$

Hence $\Delta = \Delta_1$

Remarks

(i) If Δ_1 is the determinant obtained by applying $R_i \to kR_i$ or $C_i \to kC_i$ to the determinant Δ, then $\Delta_1 = k\Delta$.

(ii) If more than one operation like $R_i \to R_i + kR_j$ is done in one step, care should be taken to see that a row that is affected in one operation should not be used in another operation. A similar remark applies to column operations.

Example 11 Prove that $\begin{vmatrix} a & a+b & a+b+c \\ 2a & 3a+2b & 4a+3b+2c \\ 3a & 6a+3b & 10a+6b+3c \end{vmatrix} = a^3$.

Solution Applying operations $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - 3R_1$ to the given determinant Δ, we have

$$\Delta = \begin{vmatrix} a & a+b & a+b+c \\ 0 & a & 2a+b \\ 0 & 3a & 7a+3b \end{vmatrix}$$

Now applying $R_3 \to R_3 - 3R_2$, we get

$$\Delta = \begin{vmatrix} a & a+b & a+b+c \\ 0 & a & 2a+b \\ 0 & 0 & a \end{vmatrix}$$

Expanding along C_1, we obtain

$$\Delta = a \begin{vmatrix} a & 2a+b \\ 0 & a \end{vmatrix} + 0 + 0$$

$$= a\,(a^2 - 0) = a\,(a^2) = a^3$$