Find $\sum_{r=1}^{n} \frac{1^3 + 2^3 + ... + r^3}{r(r+1)}$.

SOLUTION

We know that,

$1^3 + 2^3 + 3^3 + ... + n^3 = \frac{n^2(n+1)^2}{4}$

$\therefore 1^3 + 2^3 + 3^3 + ... + r^3 = \frac{r^2(r+1)^2}{4}$

$\therefore \frac{1^3 + 2^3 + 3^3 + ... + r^3}{r(r+1)} = \frac{r(r+1)}{4}$

$\therefore \sum_{r=1}^{n} \left[\frac{1^3 + 2^3 + 3^3 + ... + r^3}{r(r+1)}\right]$

$= \sum_{r=1}^{n} \frac{r(r+1)}{4}$

$= \frac{1}{4} \sum_{r=1}^{n} \left(r^2 + r\right)$

$= \frac{1}{4} \left(\sum_{r=1}^{n} r^2 + \sum_{r=1}^{n} r\right)$

$= \frac{1}{4} \left[\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}\right]$

$= \frac{1}{4} \cdot \frac{n(n+1)}{2} \left(\frac{2n+1}{3} + 1\right)$

$= \frac{n(n+1)}{8} \left(\frac{2n+1+3}{3}\right)$

$= \frac{n(n+1)(2n+4)}{24}$

$= \frac{2n(n+1)(n+2)}{24}$

$= \frac{n(n+1)(n+2)}{12}$.