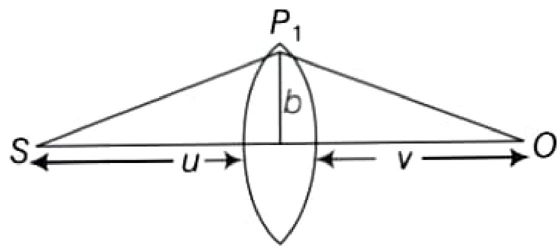
Q. (i) Consider a thin lens placed between a source (S) and an observer (O) (Figure). Let the thickness of the lens vary as $w(b) = w_0 - \frac{b^2}{\alpha}$, where b is the verticle distance from the pole, w_0 is a constant. Using Fermat's principle *i.e.*, the time of transit for a ray between the source and observer is an extremum find the condition that all paraxial rays starting from the source will converge at a point O on the axis. Find the focal length.



(ii) A gravitational lens may be assumed to have a varying width of the form

$$w(b) = k_1 \ln\left(\frac{k_2}{b}\right) \quad b_{min} < b < b_{max}$$

$$= k_1 \ln\left(\frac{k_2}{b_{min}}\right) \quad b < b_{min}$$

Show that an observer will see an image of a point object as a ring about the centre of the lens with an angular radius

$$\beta = \sqrt{\frac{(n-1)\,k_1 \frac{u}{v}}{u+v}}$$

Ans. **(i)** The time elapsed to travel from S to P_1 is

$$t_1 = \frac{SP_1}{c} = \frac{\sqrt{u^2 + b^2}}{c}$$

or $$\frac{u}{c}\left(1 + \frac{1}{2}\frac{b^2}{u^2}\right) \text{assuming } b << u_0$$

The time required to travel from P_1 to O is

$$t_2 = \frac{P_1O}{c} = \frac{\sqrt{v^2 + b^2}}{c} ; \frac{v}{c}\left(1 + \frac{1}{2}\frac{b^2}{v^2}\right)$$

The time required to travel through the lens is

$$t_1 = \frac{(n-1)\, w(b)}{c}$$

where n is the refractive index.

Thus, the total time is

$$t = \frac{1}{c} u + v + \frac{1}{2} b^2 \left(\frac{1}{u} + \frac{1}{v}\right) + (n-1)w(b)$$

Put $$\frac{1}{D} = \frac{1}{u} + \frac{1}{v}$$

Then, $$t = \frac{1}{c}\left(u + v + \frac{1}{2}\frac{b^2}{D} + (n-1)\left(w_0 + \frac{b^2}{\alpha}\right)\right)$$

Fermet's principle gives the time taken should be minimum.

For that first derivative should be zero

$$\frac{dt}{db} = 0 = \frac{b}{CD} - \frac{2(n-1)b}{c\alpha}$$

$$\alpha = 2(n-1)\, D$$

Thus, a convergent lens is formed if $\alpha = 2(n-1)D$. This is independant of and hence, all paraxial rays from S will converge at O *i.e.*, for rays

and $$(b << v)$$

Since, $\frac{1}{D} = \frac{1}{u} + \frac{1}{v}$, the focal length is D.

(ii) In this case, differentiating expression of time taken t w.r.t. b

$$t = \frac{1}{c}\left(u + v + \frac{1}{2}\frac{b^2}{D} + (n-1)\, k_1 \ln\left(\frac{k_2}{b}\right)\right)$$

$$\frac{dt}{db} = 0 = \frac{b}{D} - (n-1)\frac{k_1}{b}$$

$\Rightarrow$ $$b^2 = (n-1)\, k_1 D$$

$\therefore$ $$b = \sqrt{(n-1)k_1 D}$$

Thus, all rays passing at a height b shall contribute to the image. The ray paths make an angle.

$$\beta ; \frac{b}{v} = \frac{\sqrt{(n-1)k_1 D}}{v^2} = \sqrt{\frac{(n-1)k_1 uv}{v^2(u+v)}} = \sqrt{\frac{(n-1)k_1 u}{(u+v)v}}$$

This is the required expression.