(Question 1)

Show that the family of curves for which the slope of the tangent at any point (x, y) on it is $\frac{x^2 + y^2}{2xy}$, is given by $x^2 - y^2 = cx$.

Solution

We know that the slope of the tangent at any point on a curve is dy/dx. Therefore,

$$\frac{dy}{dx} = \frac{x^2 + y^2}{2xy}$$

or $$\frac{dy}{dx} = \frac{1 + \frac{y^2}{x^2}}{\frac{2y}{x}} \quad \text{... (1)}$$

Clearly, (1) is a homogenous differential equation. To solve it we make substitution

$$y = vx$$

Differentiating y = vx with respect to x, we get

dy / dx =v+ x (dv/ dx)

or v+ x(dv/dx) = (1+ v^2) / 2v

or x(dv/ dx) = (1- v^2) / 2v

(2v / v^2 -1)dv = dx / x

or (2v/(v^2 -1)) dv = -dx/x

Therefore ∫(2v/(v^2 -1)) = -∫ (1/x)dx

Or log | v^2 – 1 | = – log | x | + log | C1 |

or $\log |(v^2 - 1)(x)| = \log |C1|$

or $(v^2 - 1)x = \pm C1$

Replacing v by y x , we get

$$\left(\frac{y^2}{x^2} - 1\right)x = \pm C_1$$

$$(y^2 - x^2) = \pm C_1 x \text{ or } x^2 - y^2 = Cx$$