Question 2

Show that the differential equation (x – y) (dy/ dx) = x + 2y is homogeneous and solve it.

Solution

The given differential equation can be expressed as

dy / dx = (x + 2y) / (x -y) ... (1)

Let F(x, y) = (x +2y) / (x -y)

Now

$$F(\lambda x, \lambda y) = \frac{\lambda(x+2y)}{\lambda(x-y)} = \lambda^0 \cdot f(x, y)$$

Therefore, F(x, y) is a homogenous function of degree zero. So, the given differential equation is a homogenous differential equation.

Alternatively,

$$\frac{dy}{dx} = \left(\frac{1+\frac{2y}{x}}{1-\frac{y}{x}} \right) = g\left(\frac{y}{x}\right) \quad \text{... (2)}$$

R.H.S. of the differential equation (2) is of the form g (y/ x) and so it is a homogeneous function of degree zero. Therefore, equation (1) is a homogeneous differential equation.

To solve it we make the substitution

y = vx ... (3)

Differentiating equation (3) with respect to, x we get

dy /dx = v +x(dv/ dx) ... (4)

Substituting the value of y and dy /dx in equation (1) we get

v+ x(dv/ dx) = (1+ 2v) / (1 - v)

Or x(dv/ dx) =((1+ 2v) / (1 - v)) -v

or x(dv/ dx) =

$$\frac{v-1}{v^2+v+1}dv = \frac{-dx}{x}$$

Integrating both sides of equation (5), we get

$$\int \frac{v-1}{v^2+v+1}\,dv = -\int \frac{dx}{x}$$

$$\frac{1}{2}\int \frac{2v+1-3}{v^2+v+1}\,dv = -\log|x| + C_1$$

or $$\frac{1}{2}\int \frac{2v+1}{v^2+v+1}\,dv - \frac{3}{2}\int \frac{1}{v^2+v+1}\,dv = -\log|x| + C_1$$

or $$\frac{1}{2}\log\left|v^2+v+1\right| - \frac{3}{2}\int \frac{1}{\left(v+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\,dv = -\log|x| + C_1$$

or $$\frac{1}{2}\log\left|v^2+v+1\right| - \frac{3}{2}\cdot\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2v+1}{\sqrt{3}}\right) = -\log|x| + C_1$$

or $$\frac{1}{2}\log\left|v^2+v+1\right| + \frac{1}{2}\log x^2 = \sqrt{3}\tan^{-1}\left(\frac{2v+1}{\sqrt{3}}\right) + C_1$$

Replacing v by y /x , we get

$$\frac{1}{2}\log\left|\frac{y^2}{x^2}+\frac{y}{x}+1\right| = \sqrt{3}\tan^{-1}\left(\frac{x+2y}{\sqrt{3}x}\right) + C_1$$

Or 1/ 2(log I ((yy/xx)+(y/x)+1) I) $= \sqrt{3}\tan^{-1}\left(\frac{x+2y}{\sqrt{3}x}\right) + C_1$

Or 1/ 2(log I ((yy)+(yx)+xx) I)= 2 $\sqrt{3}\tan^{-1}\left(\frac{x+2y}{\sqrt{3}x}\right)$ +2 C_1

Or 1/ 2(log I ((yy)+(yx)+xx) I) =2 $\sqrt{3}\tan^{-1}\left(\frac{x+2y}{\sqrt{3}x}\right)$ + 2 C

which is the general solution of the differential equation (1)