DIFFERENTIAL EQUATIONS

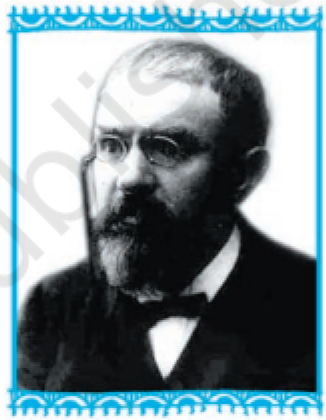
❖ *He who seeks for methods without having a definite problem in mind seeks for the most part in vain. – D. HILBERT* ❖

9.1 Introduction

In Class XI and in Chapter 5 of the present book, we discussed how to differentiate a given function f with respect to an independent variable, i.e., how to find $f'(x)$ for a given function f at each x in its domain of definition. Further, in the chapter on Integral Calculus, we discussed how to find a function f whose derivative is the function g, which may also be formulated as follows:

For a given function g, find a function f such that

$$\frac{dy}{dx} = g(x), \text{ where } y = f(x) \quad \dots (1)$$



Henri Poincare (1854-1912)

An equation of the form (1) is known as a *differential equation*. A formal definition will be given later.

These equations arise in a variety of applications, may it be in Physics, Chemistry, Biology, Anthropology, Geology, Economics etc. Hence, an indepth study of differential equations has assumed prime importance in all modern scientific investigations.

In this chapter, we will study some basic concepts related to differential equation, general and particular solutions of a differential equation, formation of differential equations, some methods to solve a first order - first degree differential equation and some applications of differential equations in different areas.

9.2 Basic Concepts

We are already familiar with the equations of the type:

$$x^2 - 3x + 3 = 0 \quad \dots (1)$$

$$\sin x + \cos x = 0 \quad \dots (2)$$

$$x + y = 7 \quad \dots (3)$$

Let us consider the equation:

$$x\frac{dy}{dx} + y = 0 \qquad \text{... (4)}$$

We see that equations (1), (2) and (3) involve independent and/or dependent variable (variables) only but equation (4) involves variables as well as derivative of the dependent variable y with respect to the independent variable x. Such an equation is called a *differential equation*.

In general, an equation involving derivative (derivatives) of the dependent variable with respect to independent variable (variables) is called a differential equation.

A differential equation involving derivatives of the dependent variable with respect to only one independent variable is called an ordinary differential equation, e.g.,

$$2\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 = 0 \text{ is an ordinary differential equation} \qquad \text{.... (5)}$$

Of course, there are differential equations involving derivatives with respect to more than one independent variables, called partial differential equations but at this stage we shall confine ourselves to the study of ordinary differential equations only. Now onward, we will use the term 'differential equation' for 'ordinary differential equation'.

Note

1. We shall prefer to use the following notations for derivatives:

   $$\frac{dy}{dx} = y', \frac{d^2y}{dx^2} = y'', \frac{d^3y}{dx^3} = y'''$$

2. For derivatives of higher order, it will be inconvenient to use so many dashes as supersuffix therefore, we use the notation y_n for nth order derivative $\frac{d^ny}{dx^n}$.

9.2.1. *Order of a differential equation*

Order of a differential equation is defined as the order of the highest order derivative of the dependent variable with respect to the independent variable involved in the given differential equation.

Consider the following differential equations:

$$\frac{dy}{dx} = e^x \qquad \text{... (6)}$$

$$\frac{d^2y}{dx^2} + y = 0 \qquad \text{... (7)}$$

$$\left(\frac{d^3y}{dx^3}\right) + x^2\left(\frac{d^2y}{dx^2}\right)^3 = 0 \qquad \text{... (8)}$$

The equations (6), (7) and (8) involve the highest derivative of first, second and third order respectively. Therefore, the order of these equations are 1, 2 and 3 respectively.

9.2.2 *Degree of a differential equation*

To study the degree of a differential equation, the key point is that the differential equation must be a polynomial equation in derivatives, i.e., y', y'', y''' etc. Consider the following differential equations:

$$\frac{d^3y}{dx^3} + 2\left(\frac{d^2y}{dx^2}\right)^2 - \frac{dy}{dx} + y = 0 \qquad \text{... (9)}$$

$$\left(\frac{dy}{dx}\right)^2 + \left(\frac{dy}{dx}\right) - \sin^2 y = 0 \qquad \text{... (10)}$$

$$\frac{dy}{dx} + \sin\left(\frac{dy}{dx}\right) = 0 \qquad \text{... (11)}$$

We observe that equation (9) is a polynomial equation in y''', y'' and y', equation (10) is a polynomial equation in y' (not a polynomial in y though). Degree of such differential equations can be defined. But equation (11) is not a polynomial equation in y' and degree of such a differential equation can not be defined.

By the degree of a differential equation, when it is a polynomial equation in derivatives, we mean the highest power (positive integral index) of the highest order derivative involved in the given differential equation.

In view of the above definition, one may observe that differential equations (6), (7), (8) and (9) each are of degree one, equation (10) is of degree two while the degree of differential equation (11) is not defined.

Note Order and degree (if defined) of a differential equation are always positive integers.

9.3. General and Particular Solutions of a Differential Equation

In earlier Classes, we have solved the equations of the type:

$$x^2 + 1 = 0 \quad \text{... (1)}$$

$$\sin^2 x - \cos x = 0 \quad \text{... (2)}$$

Solution of equations (1) and (2) are numbers, real or complex, that will satisfy the given equation i.e., when that number is substituted for the unknown x in the given equation, L.H.S. becomes equal to the R.H.S..

Now consider the differential equation $\frac{d^2y}{dx^2} + y = 0$ ... (3)

In contrast to the first two equations, the solution of this differential equation is a function ϕ that will satisfy it i.e., when the function ϕ is substituted for the unknown y (dependent variable) in the given differential equation, L.H.S. becomes equal to R.H.S..

The curve $y = \phi(x)$ is called the solution curve (integral curve) of the given differential equation. Consider the function given by

$$y = \phi(x) = a \sin(x + b), \quad \text{... (4)}$$

where $a, b \in \mathbf{R}$. When this function and its derivative are substituted in equation (3), L.H.S. = R.H.S.. So it is a solution of the differential equation (3).

Let a and b be given some particular values say $a = 2$ and $b = \frac{\pi}{4}$, then we get a function

$$y = \phi_1(x) = 2\sin\left(x + \frac{\pi}{4}\right) \quad \text{... (5)}$$

When this function and its derivative are substituted in equation (3) again L.H.S. = R.H.S.. Therefore ϕ_1 is also a solution of equation (3).

Function ϕ consists of two arbitrary constants (parameters) a, b and it is called *general solution* of the given differential equation. Whereas function ϕ_1 contains no arbitrary constants but only the particular values of the parameters a and b and hence is called a *particular solution* of the given differential equation.

The solution which contains arbitrary constants is called the *general solution* (*primitive*) of the differential equation.

The solution free from arbitrary constants i.e., the solution obtained from the general solution by giving particular values to the arbitrary constants is called a *particular solution* of the differential equation.

9.4 Formation of a Differential Equation whose General Solution is given

We know that the equation

$$x^2 + y^2 + 2x - 4y + 4 = 0 \qquad \text{... (1)}$$

represents a circle having centre at (–1, 2) and radius 1 unit.

Differentiating equation (1) with respect to x, we get

$$\frac{dy}{dx} = \frac{x+1}{2-y} \quad (y \neq 2) \qquad \text{... (2)}$$

which is a differential equation. You will find later on [See (example 9 section 9.5.1.)] that this equation represents the family of circles and one member of the family is the circle given in equation (1).

Let us consider the equation

$$x^2 + y^2 = r^2 \quad \text{... (3)}$$

By giving different values to r, we get different members of the family e.g. $x^2 + y^2 = 1$, $x^2 + y^2 = 4$, $x^2 + y^2 = 9$ etc. (see Fig 9.1). Thus, equation (3) represents a family of concentric circles centered at the origin and having different radii.

We are interested in finding a differential equation that is satisfied by each member of the family. The differential equation must be free from r because r is different for different members of the family. This equation is obtained by differentiating equation (3) with respect to x, i.e.,

$$2x + 2y\frac{dy}{dx} = 0 \quad \text{or} \quad x + y\frac{dy}{dx} = 0 \quad \text{... (4)}$$

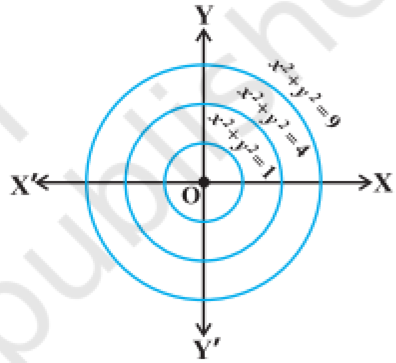


Fig 9.1

which represents the family of concentric circles given by equation (3).

Again, let us consider the equation

$$y = mx + c \quad \text{... (5)}$$

By giving different values to the parameters m and c, we get different members of the family, e.g.,

$y = x$ $\quad(m = 1, \ c = 0)$

$y = \sqrt{3}\,x$ $\quad(m = \sqrt{3}, \ c = 0)$

$y = x + 1$ $\quad(m = 1, \ c = 1)$

$y = -x$ $\quad(m = -1, \ c = 0)$

$y = -x - 1$ $\quad(m = -1, \ c = -1)$ etc. (see Fig 9.2).

Thus, equation (5) represents the family of straight lines, where m, c are parameters.

We are now interested in finding a differential equation that is satisfied by each member of the family. Further, the equation must be free from m and c because m and

c are different for different members of the family. This is obtained by differentiating equation (5) with respect to x, successively we get

$$\frac{dy}{dx} = m, \text{ and } \frac{d^2y}{dx^2} = 0 \qquad \text{... (6)}$$

The equation (6) represents the family of straight lines given by equation (5).

Note that equations (3) and (5) are the general solutions of equations (4) and (6) respectively.

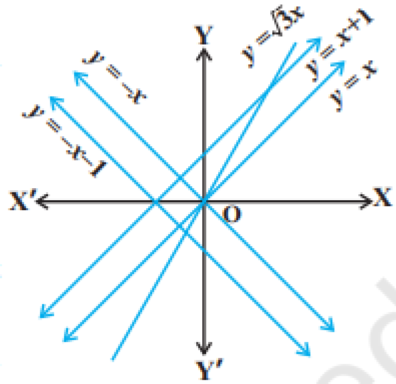


Fig 9.2

9.4.1 *Procedure to form a differential equation that will represent a given family of curves*

(a) If the given family F_1 of curves depends on only one parameter then it is represented by an equation of the form

$$F_1(x, y, a) = 0 \qquad \text{... (1)}$$

For example, the family of parabolas $y^2 = ax$ can be represented by an equation of the form $f(x, y, a) : y^2 = ax$.

Differentiating equation (1) with respect to x, we get an equation involving y', y, x, and a, i.e.,

$$g(x, y, y', a) = 0 \qquad \text{... (2)}$$

The required differential equation is then obtained by eliminating a from equations (1) and (2) as

$$F(x, y, y') = 0 \qquad \text{... (3)}$$

(b) If the given family F_2 of curves depends on the parameters a, b (say) then it is represented by an equation of the from

$$F_2(x, y, a, b) = 0 \qquad \text{... (4)}$$

Differentiating equation (4) with respect to x, we get an equation involving y', x, y, a, b, i.e.,

$$g(x, y, y', a, b) = 0 \qquad \text{... (5)}$$

But it is not possible to eliminate two parameters a and b from the two equations and so, we need a third equation. This equation is obtained by differentiating equation (5), with respect to x, to obtain a relation of the form

$$h(x, y, y', y'', a, b) = 0 \qquad \text{... (6)}$$