Question

What is the general solution of the differential equation $(2x - y + 1)\,dx + (2y - x + 1)\,dy = 0$?

Solution:

$(2x - y + 1)\, dx + (2y - x + 1)\, dy = 0$

$dy / dx = 2x - y + 1x - 2y - 1$, put $x = X + h$, $y = Y + k$

$dY / dX = [2X - Y + 2h - k + 1] / [X - 2Y + h - 2k - 1]$

$2h - k + 1 = 0$

$h - 2k - 1 = 0$

On solving $h = -1$, $k = -1$;

$dY/ dX = [2X - Y] / [X - 2Y]$

Put $Y = vX$;

$dY / dX = v + [X\, dv / dX]$

$v + [X\, dv / dX] = [2X - vX] / [X - 2vX] = [2 - v] / [1 - 2v]$

$X\, dv / dX = [2 - 2v + 2v^2] / [1 - 2v] = 2 (v^2 - v + 1) / [1 - 2v]$

$dX / X = (1 - 2v) / 2 (v^2 - v + 1)\, dv$

Put $v^2 - v + 1 = t$

$(2v - 1)\, dv = dt$

$dX / X = -dt / 2t$

$\log X = \log t^{-\frac{1}{2}} + \log c$

$X = t^{-1/2} c$

$X = (v^2 - v + 1)^{-1/2} * c$

$X^2 (v^2 - v + 1) = \text{constant}$

$(x + 1)^2 (([y + 1)^2 / (x + 1)^2] - [(y+1) / (x + 1)] + 1) = \text{constant}$

$(y + 1)^2 - (y + 1) (x + 1) + (x + 1)^2 = c$

$y^2 + x^2 - xy + x + y = c$