Prove that a triangle ABC is equilateral iff $\tan A + \tan B + \tan C = 3\sqrt{3}$.

Solution:

Let us suppose that the triangle ABC is equilateral.

Then $A = B = C = 60°$

Hence, the required value $\tan A + \tan B + \tan C$

$= \tan 60° + \tan 60° + \tan 60°$

$= 2 \tan 60°$

$= 3\sqrt{3}$

Conversely, assume that $\tan A + \tan B + \tan C = 3\sqrt{3}$

But, in triangle ABC, $A + B = 180° - C$

Taking tan on both the sides we have

$\tan (A + B) = \tan (180° - C)$

$(\tan A + \tan B)/(1 - \tan A \tan B) = - \tan$

C

Hence, this gives $\tan A + \tan B = -\tan C + \tan A \tan B \tan C$

This yields $\tan A + \tan B + \tan C = \tan A \tan B \tan C = 3\sqrt{3}$

This means that none of tan A, tan B or tan C can be negative.

Hence, ABC cannot be an obtuse angled triangle.

Moreover, we know A.M. ≥ G.M.

$1/3\ [\tan A + \tan B + \tan C] \geq [\tan A + \tan B + \tan C]1/3$

This means $1/3(3\sqrt{3}) \geq (3\sqrt{3})1/3$

This gives $\sqrt{3} \geq \sqrt{3}$.

Now, this equality can hold iff $\tan A = \tan B = \tan C$

Hence, this means iff $A = B = C$.

Hence, the triangle is equilateral.