

Que 13:

Determine the value of $\int_{-\pi}^{\pi} \frac{2x(1+\sin x)}{1+\cos^2 x} dx$.

solution:

$$\int_{-\pi}^{\pi} \frac{2x(1+\sin x)}{1+\cos^2 x} dx = I \quad (\text{say})$$

$$\text{or } I = \int_{-\pi}^{\pi} \frac{2x}{1+\cos^2 x} dx + \int_{-\pi}^{\pi} \frac{2x \sin x}{1+\cos^2 x} dx$$

$$I = 0 + 2 \int_0^{\pi} \frac{2x \sin x}{1+\cos^2 x} dx \quad \left[\because \frac{2x}{1+\cos^2 x} \text{ is an odd function} \right]$$

$$\text{or } I = 4 \int_0^{\pi} \frac{x \sin x}{1+\cos^2 x} dx \quad \dots(i)$$

$$\text{or } I = 4 \int_0^{\pi} \frac{(\pi-x) \sin(\pi-x)}{1+\{\cos(\pi-x)\}^2} dx = 4 \int_0^{\pi} \frac{(\pi-x) \sin x}{1+\cos^2 x} dx$$

$$\text{or } I = 4\pi \int_0^{\pi} \frac{\sin x}{1+\cos^2 x} dx - 1 \quad [\text{from (i)}]$$

$$\text{or } 2I = 4\pi \int_0^{\pi} \frac{\sin x}{1+\cos^2 x} dx$$

Putting $\cos x = t, -\sin x dx = dt$

When $x \rightarrow 0, t \rightarrow 1$ and when $x \rightarrow \pi, t \rightarrow -1$

$$\Rightarrow I = 2\pi \int_1^{-1} \frac{-dt}{1+t^2} = 2\pi \int_{-1}^1 \frac{dt}{1+t^2}$$

$$\because \int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx$$

$$I = 4\pi \int_0^1 \frac{dt}{1+t^2}$$

$$\Rightarrow I = 4\pi \left(\tan^{-1} t \right)_0^1 = 4\pi \{ \tan^{-1}(1) - \tan^{-1}(0) \}$$

$$\Rightarrow I = 4\pi \left\{ \frac{\pi}{4} - 0 \right\} = \pi^2$$