

Find the value of $\int_{-\pi/3}^{\pi/3} \frac{\pi + 4x^3}{2 - \cos\left(|x| + \frac{\pi}{3}\right)} dx$

solution:

$$\begin{aligned} \text{Let } I &= \int_{-\pi/3}^{\pi/3} \frac{\pi + 4x^3}{2 - \cos\left(|x| + \frac{\pi}{3}\right)} dx \\ &= \int_{-\pi/3}^{\pi/3} \frac{\pi}{2 - \cos\left(|x| + \frac{\pi}{3}\right)} dx + \int_{-\pi/3}^{\pi/3} \frac{4x^3}{2 - \cos\left(|x| + \frac{\pi}{3}\right)} dx \end{aligned}$$

The second integral becomes zero integrand being an odd function of x .

$$= 2\pi \int_0^{\pi/3} \frac{dx}{2 - \cos\left(x + \frac{\pi}{3}\right)}$$

{using the prop. of even function and also $|x| = x$ for

$$0 \leq x \leq \pi/3 \}$$

Let $x + \pi/3 = y \Rightarrow dx = dy$

also as $x \rightarrow 0, y \rightarrow \pi/3$ as $x \rightarrow \pi/3, y \rightarrow 2\pi/3$

\therefore The given integral becomes

$$= 2\pi \int_{\pi/3}^{2\pi/3} \frac{dy}{2 - \cos y} = 2\pi \int_{\pi/3}^{2\pi/3} \frac{dy}{2 - \frac{1 - \tan^2 y/2}{1 + \tan^2 y/2}}$$

$$= 2\pi \int_{\pi/3}^{2\pi/3} \frac{\sec^2 y/2}{3 \tan^2 y/2 + 1} dy$$

$$= \frac{2\pi}{3} \int_{\pi/3}^{2\pi/3} \frac{\sec^2 y/2}{\tan^2 y/2 + (1/\sqrt{3})^2} dy$$

$$= \frac{4\pi\sqrt{3}}{3} \left[\tan^{-1}(\sqrt{3} \tan y/\sqrt{2}) \right]_{\pi/3}^{2\pi/3}$$

$$= \frac{4\pi}{\sqrt{3}} \left[\tan^{-1} 3 - \pi/4 \right]$$