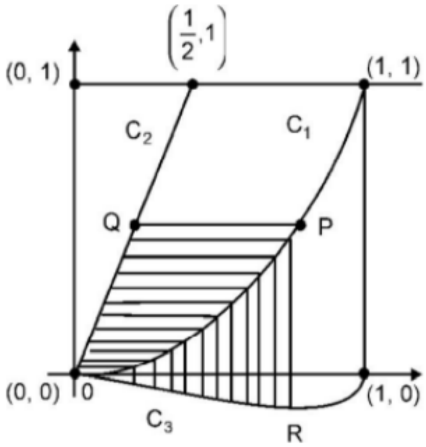
Que 12:

Let C_1 and C_2 be the graphs of the functions $y = x^2$ and $y = 2x$, $0 \le x \le 1$ respectively. Let C_3 be the graph of a function $y = f(x)$, $0 \le x \le 1$, $f(0) = 0$. For a point P on C_1, let the lines through P, parallel to the axes, meet C_2 and C_3 at Q and R respectively (see figure.) If for every position of P (on C_1), the areas of the shaded regions OPQ and ORP are equal, determine the function $f(x)$.

[1998 - 8 Marks]



solutions

$$f(x) = x^3 - x^2$$

Let P be on C_1, $y = x^2$ be (t, t^2)

$\therefore$ ordinate of Q is also t^2.

Now Q lies on $y = 2x$, and $y = t^2$

$\therefore \quad x = t^2/2$

$$\therefore \quad Q\left(\frac{t^2}{2}, t^2\right)$$

For point R, $x = t$ and it is on $y = f(x)$

$\therefore$ R is $[t, f(t)]$

$$\text{Area } OPQ = \int_0^{t^2} (\underset{C_1}{x_1} - \underset{C_2}{x_2})dy = \int_0^{t^2} \left(\sqrt{y} - \frac{y}{2}\right) dy$$

$$= \frac{2}{3}t^3 - \frac{t^4}{4} \qquad \text{...(i)}$$

$$\text{Area } OPR = \int_{0_{C_1}}^{t} ydx + \left|\int_{0_{C_3}}^{t} ydx\right|$$

$$= \int_0^t x^2 dx + \left|\int_0^t f(x)dx\right| = \frac{t^3}{3} + \left|\int_0^t f(x)dx\right| \qquad \text{....(ii)}$$

Equating (i) and (ii), we get,

$$\frac{t^3}{3} - \frac{t^4}{4}\left|\int_0^t f(x)dx\right|$$

Differentiating both sides,we get,

$$t^2 - t^3 = -f(t)$$

$$\therefore \quad f(t) = x^3 - x^2 .$$