Q8 Let $f(0,\infty) \to R$ be a differentiable function such that $f'(x) = 2 - \frac{f(x)}{x}$ for all $x \in (0,\infty)$ and $f(1) \neq 1$ Then

(A) $\lim_{x\to 0^+} f'\left(\frac{1}{x}\right) = 1$ (B) $\lim_{x\to 0^+} x f\left(\frac{1}{x}\right) = 2$

(C) $\lim_{x\to 0^+} x^2 f'(x) = 0$ (D) $|f(x)| \leq 2$ for all $x \in (0,2)$

$f'(x) = 2 - \frac{f(x)}{x} \Rightarrow f'(x) + \frac{f(x)}{x} = 2$

Its a linear differential equation. $y e^{\int \frac{1}{x} dx} = 2\int x\, dx + c$

$f(x)\, x = x^2 + c \rightarrow \boxed{f(x) = x + \frac{c}{x}}$

As $f(1) \neq 1 \Rightarrow c \neq 0$

let first see option (B) as its most easy to calculate

$x f\left(\frac{1}{x}\right) = \cancel{x}\left[\frac{1}{\cancel{x}} + \frac{cx^2}{\cancel{x}}\right] = 1 + cx^2$

$\lim_{x\to 0} x f\left(\frac{1}{x}\right) = 1 + c(0)^2 = 1$, thus option B is incorrect.

Now lets calculate $f'(x) \rightarrow f'(x) = 1 - \frac{c}{x^2}$

$\lim_{x\to 0} f'\left(\frac{1}{x}\right) = 1 - c(0)^2 = 1$ option A is correct.

$\lim_{x\to 0} x^2 f'(x) = (0)^2 - c = -c \neq 0$ option C is incorrect.

for option D, if $c > 0$ [we can take any value of c]
$f(x) \to \infty$ as $x \to 0$ thus option D is incorrect