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Chapter 9

SEQUENCES AND SERIES

❖*Natural numbers are the product of human spirit. – DEDEKIND* ❖

9.1 Introduction

Fibonacci
(1175-1250)

In mathematics, the word, "*sequence*" is used in much the same way as it is in ordinary English. When we say that a collection of objects is listed in a sequence, we usually mean that the collection is ordered in such a way that it has an identified first member, second member, third member and so on. For example, population of human beings or bacteria at different times form a sequence. The amount of money deposited in a bank, over a number of years form a sequence. Depreciated values of certain commodity occur in a sequence. Sequences have important applications in several spheres of human activities.

Sequences, following specific patterns are called *progressions*. In previous class, we have studied about *arithmetic progression* (A.P). In this Chapter, besides discussing more about A.P.; *arithmetic mean, geometric mean, relationship between A.M. and G.M., special series in forms of sum to n terms of consecutive natural numbers, sum to n terms of squares of natural numbers and sum to n terms of cubes of natural numbers* will also be studied.

9.2 Sequences

Let us consider the following examples:

Assume that there is a generation gap of 30 years, we are asked to find the number of ancestors, i.e., parents, grandparents, great grandparents, etc. that a person might have over 300 years.

Here, the total number of generations $= \dfrac{300}{30} = 10$

The number of person's ancestors for the first, second, third, …, tenth generations are 2, 4, 8, 16, 32, …, 1024. These numbers form what we call a *sequence*.

Consider the successive quotients that we obtain in the division of 10 by 3 at different steps of division. In this process we get 3,3.3,3.33,3.333, ... and so on. These quotients also form a sequence. The various numbers occurring in a sequence are called its *terms*. We denote the terms of a sequence by $a_1, a_2, a_3, \ldots, a_n, \ldots$, etc., the subscripts denote the position of the term. The n^{th} term is the number at the n^{th} position of the sequence and is denoted by a_n. The n^{th} term is also called the *general term* of the sequence.

Thus, the terms of the sequence of person's ancestors mentioned above are:

$$a_1 = 2,\ a_2 = 4,\ a_3 = 8,\ \ldots,\ a_{10} = 1024.$$

Similarly, in the example of successive quotients

$$a_1 = 3,\ a_2 = 3.3,\ a_3 = 3.33,\ \ldots,\ a_6 = 3.33333, \text{ etc.}$$

A sequence containing finite number of terms is called a *finite sequence*. For example, sequence of ancestors is a finite sequence since it contains 10 terms (a fixed number).

A sequence is called *infinite*, if it is not a finite sequence. For example, the sequence of successive quotients mentioned above is an *infinite sequence*, infinite in the sense that it never ends.

Often, it is possible to express the rule, which yields the various terms of a sequence in terms of algebraic formula. Consider for instance, the sequence of even natural numbers 2, 4, 6, …

Here $a_1 = 2 = 2 \times 1 \qquad a_2 = 4 = 2 \times 2$

$a_3 = 6 = 2 \times 3 \qquad a_4 = 8 = 2 \times 4$

....

....

$a_{23} = 46 = 2 \times 23,\ a_{24} = 48 = 2 \times 24$, and so on.

In fact, we see that the n^{th} term of this sequence can be written as $a_n = 2n$, where n is a natural number. Similarly, in the sequence of odd natural numbers 1,3,5, …, the n^{th} term is given by the formula, $a_n = 2n - 1$, where n is a natural number.

In some cases, an arrangement of numbers such as 1, 1, 2, 3, 5, 8,.. has no visible pattern, but the sequence is generated by the recurrence relation given by

$$a_1 = a_2 = 1$$

$$a_3 = a_1 + a_2$$

$$a_n = a_{n-2} + a_{n-1},\ n > 2$$

This sequence is called *Fibonacci sequence*.

In the sequence of primes 2,3,5,7,…, we find that there is no formula for the n^{th} prime. Such sequence can only be described by verbal description.

In every sequence, we should not expect that its terms will necessarily be given by a specific formula. However, we expect a theoretical scheme or a rule for generating the terms $a_1, a_2, a_3, \ldots, a_n, \ldots$ in succession.

In view of the above, *a sequence can be regarded as a function whose domain is the set of natural numbers or some subset of it. Sometimes, we use the functional notation a(n) for a_n.*

9.3 Series

Let $a_1, a_2, a_3, \ldots, a_n$, be a given sequence. Then, the expression

$$a_1 + a_2 + a_3 + , \ldots + a_n + \ldots$$

is called the *series associated with the given sequence* .The series is finite or infinite according as the given sequence is finite or infinite. Series are often represented in compact form, called *sigma notation*, using the Greek letter $\sum$ (sigma) as means of indicating the summation involved. Thus, the series $a_1 + a_2 + a_3 + \ldots + a_n$ is abbreviated as $\sum_{k=1}^{n} a_k$.

Remark When the series is used, it refers to the indicated sum not to the sum itself. For example, 1 + 3 + 5 + 7 is a finite series with four terms. When we use the phrase "*sum of a series*," we will mean the number that results from adding the terms, the sum of the series is 16.

We now consider some examples.

Example 1 Write the first three terms in each of the following sequences defined by the following:

(i) $a_n = 2n + 5$, (ii) $a_n = \frac{n-3}{4}$.

Solution (i) Here $a_n = 2n + 5$

Substituting $n = 1, 2, 3$, we get

$$a_1 = 2(1) + 5 = 7, a_2 = 9, a_3 = 11$$

Therefore, the required terms are 7, 9 and 11.

(ii) Here $a_n = \frac{n-3}{4}$. Thus, $a_1 = \frac{1-3}{4} = -\frac{1}{2}, a_2 = -\frac{1}{4}, a_3 = 0$

Hence, the first three terms are $-\frac{1}{2}, -\frac{1}{4}$ and 0.

Example 2 What is the 20^{th} term of the sequence defined by

$$a_n = (n - 1)(2 - n)(3 + n)\ ?$$

Solution Putting $n = 20$, we obtain

$$a_{20} = (20 - 1)(2 - 20)(3 + 20)$$
$$= 19 \times (-18) \times (23) = -7866.$$

Example 3 Let the sequence a_n be defined as follows:

$$a_1 = 1,\ \ a_n = a_{n-1} + 2 \text{ for } n \geq 2.$$

Find first five terms and write corresponding series.

Solution We have

$$a_1 = 1, a_2 = a_1 + 2 = 1 + 2 = 3, a_3 = a_2 + 2 = 3 + 2 = 5,$$
$$a_4 = a_3 + 2 = 5 + 2 = 7, a_5 = a_4 + 2 = 7 + 2 = 9.$$

Hence, the first five terms of the sequence are 1,3,5,7 and 9. The corresponding series is 1 + 3 + 5 + 7 + 9 +...

EXERCISE 9.1

Write the first five terms of each of the sequences in Exercises 1 to 6 whose n^{th} terms are:

1. $a_n = n(n + 2)$ **2.** $a_n = \frac{n}{n+1}$ **3.** $a_n = 2^n$

4. $a_n = \frac{2n-3}{6}$ **5.** $a_n = (-1)^{n-1}\, 5^{n+1}$ **6.** $a_n = n\frac{n^2+5}{4}$.

Find the indicated terms in each of the sequences in Exercises 7 to 10 whose n^{th} terms are:

7. $a_n = 4n - 3;\ a_{17}, a_{24}$ **8.** $a_n = \frac{n^2}{2^n};\ a_7$

9. $a_n = (-1)^{n-1}n^3;\ a_9$ **10.** $a_n = \frac{n(n-2)}{n+3};\ a_{20}$.