3) If a fair coin is tossed 10 times, find the probability of

i) exactly six heads

ii) at least six heads

iii) at most six heads

Ans: The repeated tosses of a coin are Bernoulli trials. Let X denote the number of heads in an experiment of 10 trials.

Clearly, X has the binomial distribution with $n=10$ and $p=\frac{1}{2}$

Therefore $P(X=x) = {}^{n}C_{x}\, q^{n-x} p^{x}$,

$x = 0, 1, 2, \ldots, n$

Here $n=10$, $p=\frac{1}{2}$, $q=1-p=\frac{1}{2}$

Therefore $P(X=x)$

$$= {}^{10}C_{x} \left(\frac{1}{2}\right)^{x} \cdot \left(\frac{1}{2}\right)^{10-x}$$

$$= {}^{10}C_{x} \left(\frac{1}{2}\right)^{10}$$

Now i) $P(X=6) = {}^{10}C_{6}\left(\frac{1}{2}\right)^{10} = \frac{10!}{6! \times 4!} \cdot \frac{1}{2^{10}}$

$$= \frac{105}{512}$$

ii) $P(\text{at least six heads}) = P(X \geq 6)$

$= P(X=6) + P(X=7) + P(X=8) + P(X=9)$
$+ P(X=10)$

$$= {}^{10}C_6\left(\frac{1}{2}\right)^{10} + {}^{10}C_7\left(\frac{1}{2}\right)^{10} + {}^{10}C_8\left(\frac{1}{2}\right)^{10}$$

$$+ {}^{10}C_9\left(\frac{1}{2}\right)^{10} + {}^{10}C_{10}\left(\frac{1}{2}\right)^{10}$$

$$= \left[\frac{10!}{6! \times 4!} + \frac{10!}{7! \times 3!} + \frac{10!}{8! \times 2!} + \frac{10}{9! \times 1!}\right.$$

$$\left. + \frac{10!}{10!}\right]\frac{1}{2^{10}} = \frac{193}{512}$$

iii) P(at most six heads) $= P(X \leq 6)$

$$= P(X=0) + P(X=1) + P(X=2) + P(X=3)$$

$$+ P(X=4) + P(X=5) + P(X=6)$$

$$= \left(\frac{1}{2}\right)^{10} + {}^{10}C_1\left(\frac{1}{2}\right)^{10} + {}^{10}C_2\left(\frac{1}{2}\right)^{10} + {}^{10}C_3\left(\frac{1}{2}\right)^{10}$$

$$+ {}^{10}C_4\left(\frac{1}{2}\right)^{10} + {}^{10}C_5\left(\frac{1}{2}\right)^{10} + {}^{10}C_6\left(\frac{1}{2}\right)^{10}$$

$$= \frac{848}{1024} = \frac{53}{64}$$