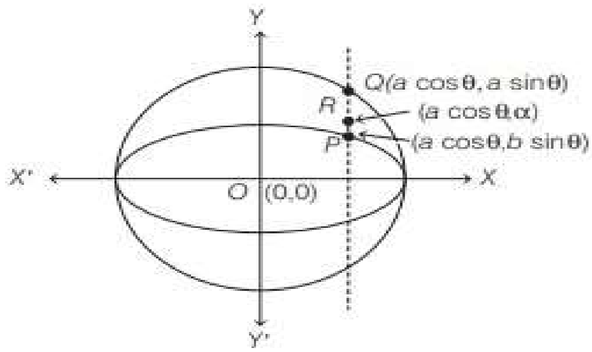
12. Let P be a point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $0<b<a$. Let the line parallel to Y-axis passing through P meet the circle $x^2+y^2=a^2$ at the point Q such that P and Q are on the same side of X-axis. For two positive real numbers r and s, find the locus of the point R on PQ such that $PR:RQ=r:s$ as P varies over the ellipse. **(2001, 4M)**

Solution: -

12. Given, $\frac{PR}{RQ}=\frac{r}{s}$



$$\Rightarrow \quad \frac{\alpha-b\sin\theta}{a\sin\theta-\alpha}=\frac{r}{s}$$

$$\Rightarrow \quad \alpha s-b\sin\theta\cdot s=ra\sin\theta-\alpha r$$

$$\Rightarrow \quad \alpha s+\alpha r=ra\sin\theta+b\sin\theta\cdot s$$

$$\Rightarrow \quad \alpha(s+r)=\sin\theta\,(ra+bs)$$

$$\Rightarrow \quad \alpha=\frac{\sin\theta\,(ra+bs)}{r+s}$$

Let the coordinates of R be (h, k).

$$\therefore \quad h=a\cos\theta \Rightarrow \cos\theta=\frac{h}{a} \quad \ldots(i)$$

and
$$k=\alpha=\frac{(ar+bs)\sin\theta}{r+s}$$

$$\Rightarrow \quad \sin\theta=\frac{k(r+s)}{ar+bs} \quad \ldots(ii)$$

On squaring and adding Eqs. (i) and (ii), we get

$$\sin^2\theta+\cos^2\theta=\frac{h^2}{a^2}+\frac{k^2(r+s)^2}{(ar+bs)^2}$$

$$\Rightarrow \quad 1=\frac{h^2}{a^2}+\frac{k^2(r+s)^2}{(ar+bs)^2}$$

Hence, locus of R is $\frac{x^2}{a^2}+\frac{y^2(r+s)^2}{(ar+bs)^2}=1$.