1. If the normal to the ellipse $3x^2 + 4y^2 = 12$ at a point P on it is parallel to the line, $2x + y = 4$ and the tangent to the ellipse at P passes through $Q(4, 4)$ then PQ is equal to

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(a) $\frac{5\sqrt{5}}{2}$ (b) $\frac{\sqrt{61}}{2}$

(c) $\frac{\sqrt{221}}{2}$ (d) $\frac{\sqrt{157}}{2}$

Solution: -

1.

> **Key Idea** Equation of tangent and normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at point $p(x_1, y_1)$ is $T = 0 \Rightarrow \frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$
> and $\frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 - b^2$ respectively.

Equation of given ellipse is $3x^2 + 4y^2 = 12$

$$\Rightarrow \quad \frac{x^2}{4} + \frac{y^2}{3} = 1 \quad \ldots \text{(i)}$$

Now, let point $P(2\cos\theta, \sqrt{3}\sin\theta)$, so equation of tangent to ellipse (i) at point P is

$$\frac{x\cos\theta}{2} + \frac{y\sin\theta}{\sqrt{3}} = 1 \quad \ldots \text{(ii)}$$

Since, tangent (ii) passes through point $Q(4, 4)$

$$\therefore 2\cos\theta + \frac{4}{\sqrt{3}}\sin\theta = 1 \quad \ldots \text{(iii)}$$

and equation of normal to ellipse (i) at point P is

$$\frac{4x}{2\cos\theta} - \frac{3y}{\sqrt{3}\sin\theta} = 4 - 3$$

$$\Rightarrow \quad 2x\sin\theta - \sqrt{3}\cos\theta y = \sin\theta\cos\theta \quad \ldots \text{(iv)}$$

Since, normal (iv) is parallel to line, $2x + y = 4$

$\therefore$ Slope of normal (iv) = slope of line, $2x + y = 4$

$$\Rightarrow \quad \frac{2}{\sqrt{3}}\tan\theta = -2 \Rightarrow \tan\theta = -\sqrt{3} \Rightarrow \theta = 120^\circ$$

$$\Rightarrow (\sin\theta, \cos\theta) = \left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$$

Hence, point $P\left(-1, \frac{3}{2}\right)$

Now, $PQ = \sqrt{(4+1)^2 + \left(4 - \frac{3}{2}\right)^2}$

[given cordinates of $Q \equiv (4, 4)$]

$$= \sqrt{25 + \frac{25}{4}} = \frac{5\sqrt{5}}{2}$$