

**Example 8** In a triangle ABC, if

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 + \sin A & 1 + \sin B & 1 + \sin C \\ \sin A + \sin^2 A & \sin B + \sin^2 B & \sin C + \sin^2 C \end{vmatrix} = 0,$$

then prove that  $\Delta ABC$  is an isosceles triangle.

**Solution** Let  $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 + \sin A & 1 + \sin B & 1 + \sin C \\ \sin A + \sin^2 A & \sin B + \sin^2 B & \sin C + \sin^2 C \end{vmatrix}$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 1 + \sin A & 1 + \sin B & 1 + \sin C \\ -\cos^2 A & -\cos^2 B & -\cos^2 C \end{vmatrix} \quad R_3 \rightarrow R_3 - R_2$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ 1 + \sin A & \sin B - \sin A & \sin C - \sin B \\ -\cos^2 A & \cos^2 A - \cos^2 B & \cos^2 B - \cos^2 C \end{vmatrix} \quad (C_3 \rightarrow C_3 - C_2 \text{ and } C_2 \rightarrow C_2 - C_1)$$

Expanding along  $R_1$ , we get

$$\begin{aligned} \Delta &= (\sin B - \sin A) (\sin^2 C - \sin^2 B) - (\sin C - \sin B) (\sin^2 B - \sin^2 A) \\ &= (\sin B - \sin A) (\sin C - \sin B) (\sin C - \sin A) = 0 \end{aligned}$$

$$\Rightarrow \text{either } \sin B - \sin A = 0 \text{ or } \sin C - \sin B \text{ or } \sin C - \sin A = 0$$

$$\Rightarrow A = B \text{ or } B = C \text{ or } C = A$$

i.e. triangle ABC is isosceles.