Find $\frac{dy}{dx}$ of the functions in, $(\cos x)^y = (\cos y)^x$

Given

$$(\cos x)^y = (\cos y)^x$$

Taking log both sides

$$\log(\cos x)^y = \log(\cos y)^x$$

$$y.\log(\cos x) = x.\log(\cos y) \qquad (As\ log(a^b) = b.log\ a)$$

Differentiating both sides $w.r.t.x.$

$$\frac{d(y.\log(\cos x))}{dx} = \frac{d(x.\log(\cos y))}{dx}$$

$$\frac{d(y \cdot \log (\cos x))}{dx}$$

Using product Rule

As $(uv)' = u'v + v'u$

$$= \frac{d(y)}{dx} \cdot \log \cos x + \frac{d(\log (\cos x))}{dx} \cdot y$$

$$= \frac{dy}{dx} \cdot \log \cos x + \frac{1}{\cos x} \cdot \frac{d(\cos x)}{dx} \cdot y$$

$$= \frac{dy}{dx} \cdot \log \cos x + \frac{1}{\cos x} \cdot (- \sin x) \cdot y$$

$$= \frac{dy}{dx} \cdot \log \cos x + \frac{(- \sin x)}{\cos x} \cdot y$$

$$= \frac{dy}{dx} \cdot \log \cos x - \tan x \cdot y$$

$$\frac{d(x.\log(\cos y))}{dx}$$

Using product Rule

As $(uv)' = u'v + v'u$

$$= \frac{d(x)}{dx} . \log\cos y + \frac{d(\log(\cos y))}{dx} . x$$

$$= \log\cos y + \frac{1}{\cos y} . \frac{d(\cos y)}{dx} . x$$

$$= \log\cos y + \frac{1}{\cos y} . \frac{d(\cos y)}{dx} . \frac{dy}{dy} . x$$

$$= \log\cos y + \frac{1}{\cos y} . \frac{d(\cos y)}{dy} . \frac{dy}{dx} . x$$

$$= \log \cos y + \frac{1}{\cos y} \cdot (-\sin y) \cdot \frac{dy}{dx} \cdot x$$

$$= \log \cos y + -\tan y \cdot x \cdot \frac{dy}{dx}$$

Now,

$$\frac{d(y \cdot \log(\cos x))}{dx} = \frac{d(x \cdot \log(\cos y))}{dx}$$

$$\frac{dy}{dx} \log \cos x - \tan x \cdot y = \log \cos y - \tan y \cdot x \cdot \frac{dy}{dx}$$

$$\frac{dy}{dx} \log \cos x - y \cdot \tan x = \log \cos y - x \cdot \tan y \cdot \frac{dy}{dx}$$

$$\frac{dy}{dx} \log \cos x + x \tan \frac{dy}{dx} = \log \cos y + y \tan x$$

$$\frac{dy}{dx} (\log \cos x + x \tan y) = \log \cos y + y \tan x$$

$$\frac{dy}{dx} = \frac{\log \cos y + y \tan x}{\log \cos x + x \tan y}$$