

56. In any ΔABC , prove that

$$\Delta = \frac{a^2 - b^2}{2} \times \frac{\sin A \sin B}{\sin(A - B)}$$

57. If the angles of triangle are 30° and 45° and the included side is $(\sqrt{3} + 1)$ cm., then prove that the area of the triangle is $\frac{1}{2}(\sqrt{3} + 1)$ cm².

58. In a ΔABC , prove that

$$\frac{\cot\left(\frac{A}{2}\right) + \cot\left(\frac{B}{2}\right) + \cot\left(\frac{C}{2}\right)}{\cot A + \cot B + \cot C} = \frac{(a + b + c)^2}{a^2 + b^2 + c^2}$$

59. If in a ΔABC , prove that $\Delta < \frac{s^2}{4}$.

60. If α, β, γ are the lengths of the altitudes of a ΔABC , then prove that

$$\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} = \frac{1}{\Delta} (\cot A + \cot B + \cot C)$$

61. If p_1, p_2, p_3 are the altitudes of a triangle from the vertices A, B, C and Δ be the area of the ΔABC , prove that

$$\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = \frac{2ab}{(a + b + c) \times \Delta} \times \cos^2\left(\frac{C}{2}\right)$$

62. If a, b, c and d are the sides of a quadrilateral, then find the minimum value of $\left(\frac{a^2 + b^2 + c^2}{d^2}\right)$

63. In a ΔABC , if $\cos A + \cos B + \cos C = \frac{3}{2}$, then the triangle is equilateral.

64. In a ΔPQR , if $\sin P, \sin Q, \sin R$ are in AP then prove that its altitudes are in HP

65. In a ΔABC , $\Delta = (6 + 2\sqrt{3})$ sq.u

and $\angle B = 45^\circ, a = 2(\sqrt{3} + 1)$, then prove that the side b is 4

66. If the angles of a triangle are 30° and 45° and the included side is $(\sqrt{3} + 1)$, then prove that

$$\text{ar}(\Delta ABC) = \frac{1}{2}(\sqrt{3} + 1)\text{sq.u}$$

67. The two adjacent sides of a cyclic quadrilateral are 2 and 3 and the angle between them is 60° . If the area of the quadrilateral is $4\sqrt{3}$, then prove that the remaining two sides are 2 and 3 respectively.

M-N THEOREM

68. The median AD of a ΔABC is perpendicular to AB . Prove that $\tan A + 2 \tan B = 0$

69. If D be the mid point of the side BC of the triangle ABC and Δ be its area, then prove that

$$\cot \theta = \frac{b^2 - c^2}{4\Delta}, \text{ where } \angle ADB = \theta$$

CIRCUM-CIRCLE AND CIRCUM-RADIUS

70. In a ΔABC , if $a = 18$ cm, $b = 24$ cm and $c = 30$ cm, then find its circum-radius

71. In an equilateral triangle of side $2\sqrt{3}$ cm, then find the circum-radius.

72. If the length of the sides of a triangle are 3, 4 and 5 units, then find its circum-radius R .

73. If $8R^2 = a^2 + b^2 + c^2$, then prove that the triangle is right angled

74. In any ΔABC , prove that $a \cos A + b \cos B + c \cos C = 4R \sin A \sin B \sin C$

75. In any ΔABC , prove that

$$D = 2R^2 \sin A \sin B \sin C$$

76. In any ΔABC , prove that,

$$\frac{\sin A}{a} + \frac{\sin B}{b} + \frac{\sin C}{c} = \frac{3}{2R}$$

77. In any ΔABC , a, b, c are in AP and p_1, p_2 and p_3 are the altitudes of the given triangle, then prove that,

$$\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} \leq \frac{3R}{\Delta}$$

78. If p_1, p_2 and p_3 are the altitudes of a ΔABC from the vertices A, B and C respectively. then prove that

$$\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = \frac{1}{r}$$

79. If p_1, p_2 and p_3 are the altitudes of a ΔABC from the vertices A, B and C respectively. then prove that

$$\frac{\cos A}{p_1} + \frac{\cos B}{p_2} + \frac{\cos C}{p_3} = \frac{1}{R}$$

80. In an acute angled ΔABC , prove that

$$\frac{\cos C}{\sqrt{4R^2 - c^2}} = \frac{1}{2R}$$

81. If p_1, p_2 and p_3 are the altitudes of a ΔABC from the vertices A, B and C respectively. then prove that

$$p_1 p_2 p_3 = \frac{a^2 b^2 c^2}{8R^3}$$

82. If p_1, p_2 and p_3 are the altitudes of a ΔABC from the vertices A, B and C respectively. then prove that

$$\frac{b p_1}{c} + \frac{c p_2}{a} + \frac{a p_3}{b} = \frac{a^2 + b^2 + c^2}{2R}$$

83. O is the circum-centre of ΔABC and R_1, R_2 and R_3 are respectively the radii of the circum-centre of the triangles $\Delta OBC, \Delta OCA$ and ΔOAB , prove that

$$\frac{a}{R_1} + \frac{b}{R_2} + \frac{c}{R_3} = \frac{abc}{R^3}$$

84. In an acute angled ΔABC , prove that

$$\frac{a \sec A + b \sec B + c \sec C}{\tan A \tan B \tan C} = 2R$$

85. In any ΔABC , prove that

$$(a \cos A + b \cos B + c \cos C) = 4R \sin A \sin B \sin C$$

IN-CIRCLE AND IN-RADIUS

86. In a ΔABC , if $a = 4$ cm, $b = 6$ cm and $c = 8$ cm, then find its in-radius.

87. If the sides of a triangle be 18, 24, 30 cm, then find its in-radius.

88. If the sides of a triangle are 3 : 7 : 8, then find $R : r$.

89. Two sides of a triangle are 2 and $\sqrt{3}$ and the included angle is 30° , then prove that its in-radius is $\frac{1}{2}(\sqrt{3} - 1)$.

90. In an equilateral triangle, prove that $R = 4r$

91. In a ΔABC , prove that

$$\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca} + \frac{1}{2rR}$$

92. In a ΔABC , prove that

$$\cos A + \cos B + \cos C = \left(1 + \frac{r}{R}\right)$$

93. In a ΔABC , prove that

$$\sin A + \sin B + \sin C = \frac{s}{R} = \frac{\Delta}{Rr}$$

94. In any ΔABC , prove that $a \cot A + b \cot B + c \cot C = 2(r + R)$

95. In a ΔABC , prove that

$$\frac{a \sec A + b \sec B + c \sec C}{2 \tan A \cdot \tan B \cdot \tan C} = R$$

96. In a ΔABC , prove that

$$(b + c) \tan \left(\frac{A}{2}\right) + (c + a) \tan \left(\frac{B}{2}\right) + (a + b) \tan \left(\frac{C}{2}\right) = 4(r + R)$$

97. In a ΔABC , if $C = 90^\circ$, prove that

$$\frac{1}{2}(a + b) = R + r$$

98. In any ΔABC , prove that

$$\cos^2 \left(\frac{A}{2}\right) + \cos^2 \left(\frac{B}{2}\right) + \cos^2 \left(\frac{C}{2}\right) = 2 + \frac{r}{2R}$$

99. If the distances of the sides of a triangle ABC from a circum-center be x, y and z respectively, then prove that

$$\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = \frac{abc}{4xyz}$$

100. If in a ΔABC , O is the circum center and R is the circum-radius and R_1, R_2, R_3 are the circum radii of the triangles $\Delta OBC, \Delta OCA$ and ΔOAB respectively, then prove that

$$\frac{a}{R_1} + \frac{b}{R_2} + \frac{c}{R_3} = \frac{abc}{R^3}$$

101. If p_1, p_2, p_3 are respectively the perpendiculars from the vertices of a Δ to the opposite sides, then prove that

$$p_1 \cdot p_2 \cdot p_3 = \frac{(abc)^2}{8R^3}$$

102. Find The bisectors of the angles of a ΔABC

EXCIRCLE AND EX-RADII

103. In a ΔABC , if $a = 18$ cm, $b = 24$ cm, and $c = 30$ cm, then find the value of r_1, r_2 and r_3

104. In a triangle ΔABC , prove that $\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = \frac{1}{r}$, where r is in radius and r_1, r_2, r_3 are exradii.

105. In a ΔABC , prove that

$$\frac{b-c}{r_1} + \frac{c-a}{r_2} + \frac{a-b}{r_3} = 0$$

106. In a ΔABC if $\frac{s-c}{s-a} = \frac{b-c}{a-b}$, then prove that a, b, c are in AP

107. In a triangle if $\left(1 - \frac{r_1}{r_2}\right)\left(1 - \frac{r_1}{r_3}\right) = 2$, prove that the triangle is right angled.

108. In a triangle ΔABC , prove that $r_1 + r_2 + r_3 - r = 4R$

109. In a triangle ΔABC , prove that $r_1 r_2 + r_2 r_3 + r_3 r_1 = s^2$

110. In a triangle ΔABC , prove that $r_1 + r_2 - r_3 + r = 4R \cos C$

111. If r_1, r_2, r_3 are in HP, then prove that a, b, c are in AP.

112. In a triangle ABC , if a, b, c are in AP as well as in GP then prove that the value of $\left(\frac{r_1}{r_2} - \frac{r_2}{r_3} + 10\right)$ is 10.

113. In a triangle ΔABC , prove that

$$\frac{1}{r_1^2} + \frac{1}{r_2^2} + \frac{1}{r_3^2} + \frac{1}{r^2} = \frac{a^2 + b^2 + c^2}{\Delta^2}$$

114. In a triangle ΔABC , prove that $(r_1 - r)(r_2 - r)(r_3 - r) = 4r^2 R$

115. If $r_1 < r_2 < r_3$ and the ex-radii of a right angled triangle and $r_1 = 1, r_2 = 2$, then prove that $r_3 = \frac{3 + \sqrt{17}}{2}$.

116. Two sides of a triangle are the roots of $x^2 - 5x + 3 = 0$. If the angle between the sides is $\frac{\pi}{3}$, then prove that the value of $r \cdot R$ is $2/3$.

117. In an isosceles triangle of which one angle is 120° , circle of radius $\sqrt{3}$ is inscribed, then prove that the area of the triangle is $(12 + 7\sqrt{3})$ sq. u.

118. If in a triangle $r = r_1 - r_2 - r_3$, then prove that the triangle is right angled.

119. In a ΔABC , prove that $r \cdot r_1 \cdot r_2 \cdot r_3 = \Delta^2$

120. Prove that $\frac{(r_1 + r_2)}{1 + \cos C} = \frac{(r_2 + r_3)}{1 + \cos A} = \frac{(r_3 + r_1)}{1 + \cos B}$

121. Prove that $\left(\frac{1}{r} - \frac{1}{r_1}\right)\left(\frac{1}{r} - \frac{1}{r_2}\right)\left(\frac{1}{r} - \frac{1}{r_3}\right) = \frac{16R}{r^2(a+b+c)^2}$

122. Prove that $\left(\frac{1}{r_1} + \frac{1}{r_2}\right)\left(\frac{1}{r_2} + \frac{1}{r_3}\right)\left(\frac{1}{r_3} + \frac{1}{r_1}\right) = \frac{64R^3}{(abc)^2}$

123. In a ΔABC , prove that

$$r^2 + r_1^2 + r_2^2 + r_3^2 = 16R^2 - (a^2 + b^2 + c^2)$$

124. In a ΔABC , prove that

$$\frac{(r_1 + r_2)(r_2 + r_3)(r_3 + r_1)}{(r_1 r_2 + r_2 r_3 + r_3 r_1)} = 4R$$

REGULAR POLYGON

125. If A_0, A_1, \dots, A_5 be the consecutive vertices of a regular hexagon inscribed in a unit circle. Then find the product of length of A_0A_1, A_0A_2 and A_0A_4 .

126. If the Area of circle is A_1 and area of regular pentagon inscribed in the circle is A_2 . Find the ratio of area of two.

127. Let A_1, A_2, A_3, A_4 and A_5 be the vertices of a regular pentagon inscribed in a unit circle taken in order. Show that $A_1A_2 \times A_1A_3 = \sqrt{5}$.

128. The sides of a regular do-decagon is 2 ft. Find the radius of the circumscribed circle.

129. A regular pentagon and a regular decagon have the same perimeter. Find the ratio of its area.

130. If $2a$ be the sides of a regular polygon of n -sides. R and r be the circum-radius and inradius, then prove that

$$r + R = a \cot\left(\frac{\pi}{2n}\right).$$

131. A regular pentagon and a regular decagon have the same area, then find the ratio of their perimeter.

132. If the number of sides of two regular polygon having the same perimeter be n and $2n$ respectively, prove that their areas are in the ratio

$$2 \cos\left(\frac{\pi}{n}\right) : \left(1 + \cos\left(\frac{\pi}{n}\right)\right)$$

133. Let $A_1, A_2, A_3, \dots, A_n$ be the vertices of an n -sided regular polygon such that $\frac{1}{A_1A_2} = \frac{1}{A_1A_3} + \frac{1}{A_1A_4}$, then find the value of n .

134. If A, A_1, A_2, A_3 are the areas of incircle and the ex-circles of a triangle, then prove that $\frac{1}{\sqrt{A_1}} + \frac{1}{\sqrt{A_2}} + \frac{1}{\sqrt{A_3}} = \frac{1}{\sqrt{A}}$.

135. If the perimeter of the circle and the perimeter of the polygon of n -sides are same, then prove that the ratio of the area of the circle and the area of the polygon of

$$n\text{-sides is } \tan\left(\frac{\pi}{n}\right) : \frac{\pi}{n}$$

136. Prove that the sum of the radii of the circle, which are respectively inscribed in and circum-scribed about a

$$\text{regular polygon of } n\text{-sides, is } \frac{a}{2} \cot\left(\frac{\pi}{2n}\right)$$

$$\begin{aligned} &= \frac{k^2 \times \sin C \times \frac{a}{k} \times \frac{b}{k}}{2} \\ &= \frac{1}{2} \times ab \sin C \\ &= \Delta \end{aligned}$$

57. As we know that, $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

$$\Rightarrow \frac{(\sqrt{3}+1)}{\sin(105^\circ)} = \frac{b}{\sin 45^\circ} = \frac{c}{\sin 30^\circ}$$

$$\Rightarrow \frac{(\sqrt{3}+1)}{\cos(15^\circ)} = \frac{b}{\sin 45^\circ} = \frac{c}{\sin 30^\circ}$$

$$\Rightarrow \frac{(\sqrt{3}+1)}{\sqrt{3}+1} = \frac{b}{\frac{1}{\sqrt{2}}} = \frac{c}{\frac{1}{2}}$$

$$\Rightarrow 2\sqrt{2} = b\sqrt{2} = 2c$$

$$\Rightarrow b = 2 \text{ and } c = \sqrt{2}$$

Hence, the area of the triangle is $= \frac{1}{2}bc \sin A$

$$= \frac{1}{2} \times 2 \times \sqrt{2} \times \sin(105^\circ)$$

$$= \frac{1}{2} \times 2 \times \sqrt{2} \times \frac{\sqrt{3}+1}{2\sqrt{2}}$$

$$= \frac{(\sqrt{3}+1)}{2} \text{ s.u.}$$

58. We have

$$\cot A + \cot B + \cot C$$

$$= \frac{\cos A}{\sin A} + \frac{\cos B}{\sin B} + \frac{\cos C}{\sin C}$$

$$= \frac{b^2+c^2-a^2}{2bc} + \frac{c^2+a^2-b^2}{2ac} + \frac{a^2+b^2-c^2}{2ab}$$

$$= \frac{b^2+c^2-a^2}{2abck} + \frac{c^2+a^2-b^2}{2abck} + \frac{a^2+b^2-c^2}{2abck}$$

$$= \frac{a^2+b^2+c^2}{2abck}$$

$$= \frac{(a^2+b^2+c^2)}{4 \times \left(\frac{1}{2}ab\right) \times \sin C}$$

$$= \frac{(a^2+b^2+c^2)}{4 \times \Delta} \quad \dots(i)$$

Also, $\cot\left(\frac{A}{2}\right) + \cot\left(\frac{B}{2}\right) + \cot\left(\frac{C}{2}\right)$

$$= \frac{(a+b+c)}{(b+c-a)} \times \cot\left(\frac{A}{2}\right)$$

$$= \frac{s^2}{\Delta} \quad \dots(ii)$$

Dividing (ii) by (i), we get

$$\frac{\cot\left(\frac{A}{2}\right) + \cot\left(\frac{B}{2}\right) + \cot\left(\frac{C}{2}\right)}{\cot(A) + \cot(B) + \cot(C)} = \frac{s^2}{4(a^2+b^2+c^2)}$$

$$\Rightarrow \frac{\cot\left(\frac{A}{2}\right) + \cot\left(\frac{B}{2}\right) + \cot\left(\frac{C}{2}\right)}{\cot(A) + \cot(B) + \cot(C)} = \frac{(a+b+c)^2}{(a^2+b^2+c^2)}$$

59. Let $a, b,$ and c are the sides of a triangle and s be the semi perimeter.

Let the four quantities are $s, (s-a), (s-b)$ and $(s-c)$

Applying, $AM \geq GM$, we get

$$\Rightarrow \frac{s + (s-a) + (s-b) + (s-c)}{4}$$

$$\geq \sqrt[4]{s(s-a)(s-b)(s-c)}$$

$$\Rightarrow \frac{4s - (a+b+c)}{4} \geq \sqrt[4]{\Delta^2}$$

$$\Rightarrow \frac{4s - 2s}{4} \geq (\Delta)^{\frac{1}{2}}$$

$$\Rightarrow \frac{s}{2} \geq (\Delta)^{\frac{1}{2}}$$

$$\Rightarrow \Delta < \frac{s^2}{4}$$

60. Let $AD = \alpha, BE = \beta$ and $CF = \gamma$

$$\text{Then, } \Delta = \frac{1}{2} \times a \times AD = \frac{1}{2} \times b \times BE = \frac{1}{2} \times c \times CF$$

$$\Rightarrow AD = \frac{2\Delta}{a}, BE = \frac{2\Delta}{b}, CF = \frac{2\Delta}{c}$$

$$\Rightarrow \alpha = \frac{2\Delta}{a}, \beta = \frac{2\Delta}{b}, \gamma = \frac{2\Delta}{c}$$

$$\text{Now, } \frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2}$$

$$= \frac{a^2}{4\Delta^2} + \frac{b^2}{4\Delta^2} + \frac{c^2}{4\Delta^2}$$

$$= \frac{(a^2+b^2+c^2)}{4\Delta^2}$$

$$= \frac{1}{\Delta} \times \frac{(a^2+b^2+c^2)}{4\Delta}$$

$$= \frac{1}{\Delta} \times (\cot A + \cot B + \cot C)$$

$$= \frac{(\cot A + \cot B + \cot C)}{\Delta}$$

Hence, the result.

61. Let $AD = p_1, BE = p_2$ and $CF = p_3$

$$\text{Then, } \Delta = \frac{1}{2} \times a \times p_1 = \frac{1}{2} \times b \times p_2 = \frac{1}{2} \times c \times p_3$$

$$\Rightarrow p_1 = \frac{2\Delta}{a}, p_2 = \frac{2\Delta}{b}, p_3 = \frac{2\Delta}{c}$$

Now,

$$\begin{aligned} \frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p_3} &= \frac{a}{2\Delta} + \frac{b}{2\Delta} - \frac{c}{2\Delta} \\ &= \frac{(a+b-c)}{2\Delta} \\ &= \frac{(a+b+c-2c)}{2\Delta} \\ &= \frac{(2s-2c)}{2\Delta} \\ &= \frac{(s-c)}{\Delta} \\ &= \frac{2ab \times s(s-c)}{\Delta \times s} \times \frac{1}{2ab} \\ &= \frac{2ab}{\Delta \times s} \times \frac{s(s-c)}{2ab} \\ &= \frac{2ab}{(a+b+c)\Delta} \times \cos^2\left(\frac{C}{2}\right) \end{aligned}$$

62. We have

$$\begin{aligned} (a-b)^2 + (b-c)^2 + (c-d)^2 &\geq 0 \\ \Rightarrow 2(a^2 + b^2 + c^2) &\geq 2(ab + bc + ca) \\ \Rightarrow 3(a^2 + b^2 + c^2) &\geq (a^2 + b^2 + c^2) + (2ab + bc + ca) \\ \Rightarrow 3(a^2 + b^2 + c^2) &> (a+b+c)^2 > d^2 \\ \Rightarrow \frac{3(a^2 + b^2 + c^2)}{d^2} &> 1 \\ \Rightarrow \frac{(a^2 + b^2 + c^2)}{d^2} &> \frac{1}{3} \end{aligned}$$

Thus, the minimum value of $\frac{(a^2 + b^2 + c^2)}{d^2}$ is $\frac{1}{3}$

63. We have

$$\begin{aligned} \cos A + \cos B + \cos C &= \frac{3}{2} \\ \Rightarrow 1 + 4 \sin\left(\frac{A}{2}\right) \sin\left(\frac{B}{2}\right) \sin\left(\frac{C}{2}\right) &= \frac{3}{2} \\ \Rightarrow \sin\left(\frac{A}{2}\right) \sin\left(\frac{B}{2}\right) \sin\left(\frac{C}{2}\right) &= \frac{1}{8} \end{aligned}$$

It is possible only when

$$\begin{aligned} \sin\left(\frac{A}{2}\right) = \frac{1}{2}, \sin\left(\frac{B}{2}\right) = \frac{1}{2}, \sin\left(\frac{C}{2}\right) &= \frac{1}{2} \\ \Rightarrow \frac{A}{2} = \frac{\pi}{6}, \frac{B}{2} = \frac{\pi}{6}, \frac{C}{2} = \frac{\pi}{6} \\ \Rightarrow A = \frac{\pi}{3}, B = \frac{\pi}{3}, C = \frac{\pi}{3} \\ \Rightarrow \Delta \text{ is an equilateral.} \end{aligned}$$

64. Here,

$$\begin{aligned} \Delta &= \frac{1}{2} \times p \times p_1 = \frac{1}{2} \times q \times p_2 = \frac{1}{2} \times r \times p_3 \\ \Rightarrow p &= \frac{2\Delta}{p_1}, q = \frac{2\Delta}{p_2}, r = \frac{2\Delta}{p_3} \end{aligned}$$

From sine rule of a triangle,

$$\frac{\sin P}{p} = \frac{\sin Q}{q} = \frac{\sin R}{r}$$

Given $\sin P, \sin Q, \sin R$ are in AP

$$\begin{aligned} \Rightarrow p, q, r &\in \text{AP} \\ \Rightarrow \frac{2\Delta}{p_1}, \frac{2\Delta}{p_2}, \frac{2\Delta}{p_3} &\in \text{AP} \end{aligned}$$

65. In ΔABC , $\Delta = \frac{1}{2} \times ac \sin(\angle B)$

$$\begin{aligned} \Rightarrow (6 + 2\sqrt{3}) &= \frac{1}{2} \times 2(\sqrt{3} + 1) \times c \times \frac{1}{\sqrt{2}} \\ \Rightarrow c &= \frac{\sqrt{2}(6 + 2\sqrt{3})}{(\sqrt{3} + 1)} \\ \Rightarrow c &= \frac{2\sqrt{3}\sqrt{2}(\sqrt{3} + 1)}{(\sqrt{3} + 1)} \\ \Rightarrow c &= 2\sqrt{6} \end{aligned}$$

$$\text{Now, } \cos B = \frac{a^2 + c^2 - b^2}{2ac}$$

$$\begin{aligned} \Rightarrow \frac{1}{\sqrt{2}} &= \frac{4(\sqrt{3} + 1)^2 + 24 - b^2}{2 \cdot 2(\sqrt{3} + 1) \cdot 2\sqrt{6}} \\ \Rightarrow 4(\sqrt{3} + 1)^2 + 24 - b^2 &= 8\sqrt{3}(\sqrt{3} + 1) \\ \Rightarrow 4(4 + 2\sqrt{3}) + 24 - b^2 &= 8(3 + \sqrt{3}) \\ \Rightarrow b^2 &= 16 \\ \Rightarrow b &= 4 \end{aligned}$$

66. Let $\angle B = 30^\circ, \angle C = 45^\circ$

So, $\angle A = 180^\circ - (30^\circ + 45^\circ) = 105^\circ$

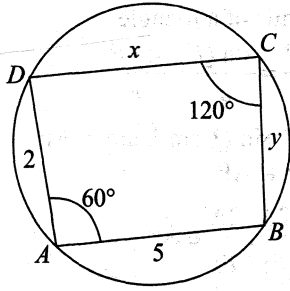
From sine formula, we can write

$$\begin{aligned} \frac{a}{\sin(105^\circ)} &= \frac{b}{\sin(30^\circ)} = \frac{c}{\sin(45^\circ)} \\ \frac{(\sqrt{3} + 1)}{\left(\frac{2\sqrt{2}}{2}\right)} &= \frac{b}{\frac{1}{2}} = \frac{c}{\frac{1}{\sqrt{2}}} \\ 2\sqrt{2} &= 2b = c\sqrt{2} \\ b &= \sqrt{2}, c = 2 \end{aligned}$$

Thus, area of triangle ABC is

$$\begin{aligned} &= \frac{1}{2} bc \sin A \\ &= \frac{1}{2} \times 2 \times \sqrt{2} \times \sin(105^\circ) \\ &= \frac{1}{2} \times 2 \times \sqrt{2} \times \left(\frac{\sqrt{3} + 1}{2\sqrt{2}}\right) \\ &= \left(\frac{\sqrt{3} + 1}{2}\right) \text{ s.u.} \end{aligned}$$

67. Suppose, $AC = 2, AB = 5, BC = x, CD = y$ and $\angle BAD = 60^\circ$



$$\begin{aligned} \text{Area of } \triangle ABC &= \frac{1}{2} \cdot 5 \cdot 2 \cdot \sin(60^\circ) \\ &= \frac{5\sqrt{3}}{2} \end{aligned}$$

Also, from $\triangle ABC$,

$$\begin{aligned} \cos(60^\circ) &= \frac{25 + 4 - BD^2}{2 \cdot 5 \cdot 2} \\ &= \frac{29 - BD^2}{20} \end{aligned}$$

$$\Rightarrow \frac{29 - BD^2}{20} = \frac{1}{2}$$

$$\Rightarrow BD^2 = 19$$

$$\Rightarrow BD = \sqrt{19}$$

Since A, B, C, D are concyclic, so $\angle BCD = 180^\circ - 60^\circ = 120^\circ$

Then, from $\triangle BCD$,

$$\cos(120^\circ) = \frac{x^2 + y^2 - (\sqrt{19})^2}{2xy}$$

$$\Rightarrow \frac{x^2 + y^2 - (\sqrt{19})^2}{2xy} = -\frac{1}{2}$$

$$\Rightarrow \frac{x^2 + y^2 - (\sqrt{19})^2}{xy} = -1$$

$$\Rightarrow x^2 + y^2 + xy = 19 \quad \dots(i)$$

Again, area of $\triangle BCD$

$$= \frac{1}{2} \cdot x \cdot y \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{3}xy}{4}$$

Thus, area of quad. $ABCD = 4\sqrt{3}$

$$\Rightarrow \frac{5\sqrt{3}}{2} + \frac{\sqrt{3}xy}{4} = 4\sqrt{3}$$

$$\Rightarrow \frac{5}{2} + \frac{xy}{4} = 4$$

$$\Rightarrow \frac{xy}{4} = 4 - \frac{5}{2} = \frac{3}{2}$$

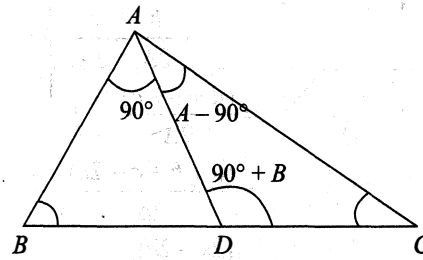
$$\Rightarrow xy = 6$$

From (i), we get

$$x^2 + y^2 = 13$$

$$\Rightarrow x = 3, y = 2$$

68. Since AD is the median, so $BD : DC = 1 : 1$



Clearly, $\angle ADC = 90^\circ + B$.

Now, applying $m : n$ rule, we get,

$$(1 + 1) \cot(90^\circ + B) = 1 \cdot \cot(90^\circ) - 1 \cdot \cot(A - 90^\circ)$$

$$\Rightarrow -2 \tan B = 0 - (-\tan A)$$

$$\Rightarrow -2 \tan B = \tan A$$

$$\Rightarrow \tan A + 2 \tan B = 0$$

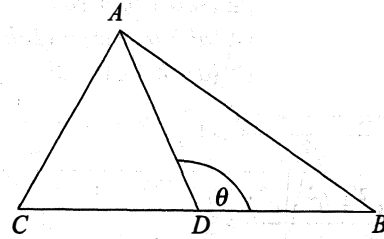
Hence, the result.

$$\Rightarrow \frac{1}{p_1}, \frac{1}{p_2}, \frac{1}{p_3} \in AP$$

$$\Rightarrow p_1, p_2, p_3 \in HP$$

Thus, the altitudes are in HP

69.



By $m : n$ rule, we get

$$(1 + 1) \cot \theta = 1 \cdot \cot C - 1 \cdot \cot B$$

$$\Rightarrow 2 \cot \theta = \cot C - \cot B$$

$$\Rightarrow 2 \cot \theta = \frac{a^2 + b^2 - c^2}{2ab \sin C} - \frac{a^2 + c^2 - b^2}{2ab \sin B}$$

$$\Rightarrow 2 \cot \theta = \frac{a^2 + b^2 - c^2}{4\Delta} - \frac{a^2 + c^2 - b^2}{4\Delta}$$

$$\Rightarrow 2 \cot \theta = \frac{2(b^2 - c^2)}{4\Delta}$$

$$\Rightarrow \cot \theta = \frac{(b^2 - c^2)}{4\Delta}$$

Hence, the result.

70. Clearly, the triangle is right angled.

$$(\because 18^2 + 24^2 = 30^2)$$

Thus, the area of the triangle

$$= \frac{1}{2} \times 24 \times 18 = 12 \times 18$$

Therefore, the circum-radius

$$= R$$

$$= \frac{abc}{4\Delta}$$

$$= \frac{18 \times 24 \times 30}{4 \times 12 \times 18} = 15$$

71. As we know that,

$$\frac{a}{\sin A} = 2R$$

$$\Rightarrow 2R = \frac{2\sqrt{3}}{\sin(60^\circ)}$$

$$\Rightarrow R = \frac{\sqrt{3}}{\frac{\sqrt{3}}{2}} = 2$$

Hence, the circum-radius is 2.

72. Let $a = 3, b = 4$ and $c = 5$

Clearly, it is a right angled triangle

$$\text{Thus, } \Delta = \frac{1}{2} \times 4 \times 3 = 6 \text{ sq.u}$$

Hence, the circum-radius R

$$\begin{aligned} &= \frac{abc}{4\Delta} \\ &= \frac{3 \times 4 \times 5}{6} \\ &= 10 \end{aligned}$$

73. We have

$$8R^2 = a^2 + b^2 + c^2 \quad 8R^2 = (2R \sin A)^2 + (2R \sin B)^2 + (2R \sin C)^2$$

$$\Rightarrow \sin^2 A + \sin^2 B + \sin^2 C = 2$$

$$\Rightarrow 1 - \cos^2 A + 1 - \cos^2 B + \sin^2 C = 2$$

$$\Rightarrow \cos^2 A - \sin^2 C + \cos^2 B = 0$$

$$\Rightarrow \cos(A+C) \cos(A-C) + \cos^2 B - 0$$

$$\Rightarrow \cos(\pi - B) \cos(A-C) + \cos^2 B = 0$$

$$\Rightarrow \cos B \cos(A-C) - \cos^2 B = 0$$

$$\Rightarrow \cos B (\cos(A-C) - \cos B) = 0$$

$$\Rightarrow \cos B (\cos(A-C) + \cos(A+C)) = 0$$

$$\Rightarrow \cos B \cdot 2 \cos A \cos C = 0$$

$$\Rightarrow \cos A = 0, \cos B = 0, \cos C = 0$$

$$\Rightarrow A = \frac{\pi}{2} \text{ or } B = \frac{\pi}{2} \text{ or } C = \frac{\pi}{2}$$

Thus, the triangle is right angled.

74. We have $a \cos A + b \cos B + c \cos C$

$$= 2R(\sin A \cdot \cos A + \sin B \cdot \cos B + \sin C \cdot \cos C)$$

$$= \frac{2R}{2} [2 \sin A \cdot \cos A + 2 \sin B \cdot \cos B + 2 \sin C \cdot \cos C]$$

$$= R(\sin 2A + \sin 2B + \sin 2C)$$

$$= R(4 \sin A \cdot \sin B \cdot \sin C)$$

$$= 4R \sin A \cdot \sin B \cdot \sin C$$

75. We have

$$\Delta = \frac{1}{2} \times a \times b \times \sin C$$

$$\Rightarrow \Delta = \frac{1}{2} \times 2R \sin A \times 2R \sin B \times \sin C$$

$$\Rightarrow \Delta = 2R^2 \cdot \sin A \cdot \sin B \cdot \sin C$$

76. We have

$$\begin{aligned} &\frac{\sin A}{a} + \frac{\sin B}{b} + \frac{\sin C}{c} \\ &= \frac{\sin A}{2R \sin A} + \frac{\sin B}{2R \sin B} + \frac{\sin C}{2R \sin C} \\ &= \frac{1}{2R} + \frac{1}{2R} + \frac{1}{2R} \\ &= \frac{3}{2R} \end{aligned}$$

77. Let $AD = p_1, BE = p_2$ and $CF = p_3$.

Then,

$$\Delta = \frac{1}{2} \times a \times p_1 = \frac{1}{2} \times b \times p_2 = \frac{1}{2} \times c \times p_3$$

$$\Rightarrow p_1 = \frac{2\Delta}{a}, p_2 = \frac{2\Delta}{b}, p_3 = \frac{2\Delta}{c}$$

Now,

$$\begin{aligned} \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} &= \frac{a+b+c}{2\Delta} \\ &= \frac{2R(\sin A + \sin B + \sin C)}{2\Delta} \\ &\leq \frac{3R}{\Delta} \quad (\because \sin A \leq 1, \sin B \leq 1, \sin C \leq 1) \end{aligned}$$

78. Here, $\Delta = \frac{1}{2}ap_1 = \frac{1}{2}bp_2 = \frac{1}{2}cp_3$

$$\Rightarrow p_1 = \frac{2\Delta}{a}, p_2 = \frac{2\Delta}{b}, p_3 = \frac{2\Delta}{c}$$

Now,

$$\begin{aligned} \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} &= \frac{a+b+c}{2\Delta} \\ &= \frac{2s}{2\Delta} \\ &= \frac{s}{\Delta} \\ &= \frac{1}{r} \end{aligned}$$

79. We have

$$\begin{aligned} &\frac{\cos A}{p_1} + \frac{\cos B}{p_2} + \frac{\cos C}{p_3} \\ &= \frac{1}{2\Delta} (a \cos A + b \cos B + c \cos C) \\ &= \frac{1}{2\Delta} [2R \sin A \cos A + 2R \sin B \cos B + 2R \sin C \cos C] \end{aligned}$$

$$\begin{aligned}
 &= \frac{R}{2\Delta}(\sin 2A + \sin 2B + \sin 2C) \\
 &= \frac{R}{2\Delta}(4 \sin A \sin B \sin C) \\
 &= \frac{2R}{\Delta}(\sin A \sin B \sin C) \\
 &= \frac{2R}{\Delta} \times \left(\frac{a}{2R}\right) \times \left(\frac{b}{2R}\right) \times (\sin C) \\
 &= \frac{1}{\Delta R} \left(\frac{1}{2} ab \sin C\right) \\
 &= \frac{1}{\Delta R} \times \Delta \\
 &= \frac{1}{R}
 \end{aligned}$$

80. We have

$$\begin{aligned}
 \frac{\cos C}{\sqrt{4R^2 - c^2}} &= \frac{\cos C}{\sqrt{4R^2 - 4R^2 \sin^2 C}} \\
 &= \frac{\cos C}{\sqrt{4R^2(1 - \sin^2 C)}} \\
 &= \frac{\cos C}{2R \cos C} = \frac{1}{2R}
 \end{aligned}$$

81. We have

$$\begin{aligned}
 \Delta &= \frac{1}{2} \times p_1 \times a \\
 p_1 &= \frac{2\Delta}{a}
 \end{aligned}$$

Similarly,

$$p_2 = \frac{2\Delta}{b}, p_3 = \frac{2\Delta}{c}$$

Now,

$$\begin{aligned}
 p_1 p_2 p_3 &= \frac{8\Delta^3}{abc} \\
 &= \frac{8 \left(\frac{abc}{4R}\right)^3}{abc} \\
 &= \frac{a^2 b^2 c^2}{8R^3}
 \end{aligned}$$

82. We have

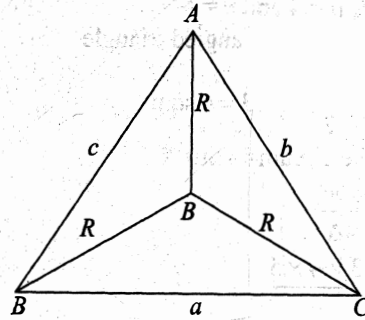
$$\begin{aligned}
 \Delta &= \frac{1}{2} \times p_1 \times a \\
 p_1 &= \frac{2\Delta}{a}
 \end{aligned}$$

Similarly, $p_2 = \frac{2\Delta}{b}, p_3 = \frac{2\Delta}{c}$

$$\text{Now, } \frac{bp_1}{c} + \frac{cp_2}{a} + \frac{ap_3}{b}$$

$$\begin{aligned}
 &= \frac{2\Delta b}{ac} + \frac{2\Delta c}{ab} + \frac{2\Delta a}{bc} \\
 &= 2\Delta \left(\frac{b}{ac} + \frac{c}{ab} + \frac{a}{bc}\right) \\
 &= \frac{2\Delta(b^2 + c^2 + a^2)}{abc} \\
 &= \frac{(a^2 + b^2 + c^2)}{2R}
 \end{aligned}$$

83. Given, O is the circumcentre of ΔABC



Let, $\text{ar}(\Delta BOC) = \Delta_1$, $\text{ar}(\Delta AOC) = \Delta_2$
and $\text{ar}(\Delta AOB) = \Delta_3$, respectively.

$$\text{Now, } R_1 = \frac{OB \cdot OC \cdot a}{4(\Delta BOC)} = \frac{aR^2}{\Delta_1}$$

$$\frac{a}{R_1} = \frac{4\Delta_1}{R^2}$$

$$\text{Similarly, } \frac{b}{R_2} = \frac{4\Delta_2}{R^2}, \frac{c}{R_3} = \frac{4\Delta_3}{R^2}$$

$$\text{Now, } \frac{a}{R_1} + \frac{b}{R_2} + \frac{c}{R_3}$$

$$= \frac{4}{R^2} (\Delta_1 + \Delta_2 + \Delta_3)$$

$$= \frac{4\Delta}{R^2}$$

$$= \frac{4}{R^2} \times \frac{abc}{4R} = \frac{abc}{R^3}$$

$$\frac{a \sec A + b \sec B + c \sec C}{\tan A \tan B \tan C} = 2$$

84. We have

$$\frac{a \sec A + b \sec B + c \sec C}{\tan A \tan B \tan C}$$

$$= \frac{2R \sin A \sec A + 2R \sin B \sec B + 2R \sin C \sec C}{\tan A \tan B \tan C}$$

$$= \frac{2R \tan A + 2R \tan B + 2R \tan C}{\tan A \tan B \tan C}$$

$$= \frac{2R(\tan A + \tan B + \tan C)}{\tan A \tan B \tan C}$$

$$= \frac{2R(\tan A \cdot \tan B \cdot \tan C)}{\tan A \tan B \tan C}$$

$$= 2R$$

85. We have

$$a \cos A + b \cos B + c \cos C$$

$$= 2R(\sin A \cos A + \sin B \cdot \cos B + \sin C \cdot \cos C)$$

$$= R(2 \sin A \cos A + 2 \sin B \cdot \cos B + 2 \sin C \cdot \cos C)$$

$$= R(\sin 2A + \sin 2B + \sin 2C)$$

$$= R(4 \sin A \cdot \sin B \cdot \sin C)$$

$$= 4(R \sin A \cdot \sin B \cdot \sin C)$$

86. Here, $s = \frac{4+6+8}{2} = 9$

Area of a triangle

$$= \sqrt{s(s-a)(s-b)(s-c)}$$

$$= \sqrt{9(9-4)(9-6)(9-8)}$$

$$= \sqrt{9 \times 5 \times 3 \times 1} = 3\sqrt{15}$$

Hence, the in-radius

$$= r = \frac{\Delta}{s} = \frac{3\sqrt{15}}{9} = \frac{\sqrt{15}}{3}$$

87. Clearly, it is a right angled triangle

So, its area = $\frac{1}{2} \times 18 \times 24 = 9 \times 24$

and $s = \frac{18+24+30}{2} = \frac{72}{2} = 36$

Thus, in-radius = $r = \frac{\Delta}{s} = \frac{9 \times 24}{36} = 6$

88. Do yourself.

89. Two sides of a triangle are 2 and $\sqrt{3}$ and the included angle is 30° , then prove that its in-radius is $\frac{1}{2}(\sqrt{3}-1)$.

89. We have area of the triangle

$$= \Delta = \frac{1}{2} \times 2 \times \sqrt{3} \times \sin(30^\circ)$$

$$= \frac{\sqrt{3}}{2}$$

Also, $a^2 = b^2 + c^2 - 2bc \cos A$

$$a^2 = 4 + 3 - 2 \times 2 \times \sqrt{3} \times \cos(30^\circ)$$

$$a^2 = 7 - 2 \times 2 \times \sqrt{3} \times \frac{\sqrt{3}}{2}$$

$$a^2 = 7 - 6 = 1$$

$$a = 1$$

Now, $s = \frac{a+b+c}{2} = \frac{1+2+\sqrt{3}}{2} = \frac{3+\sqrt{3}}{2}$

Hence, in-radius

$$= r = \frac{\Delta}{s}$$

$$= \frac{\frac{\sqrt{3}}{2}}{\frac{3+\sqrt{3}}{2}} = \frac{\sqrt{3}}{(\sqrt{3}+3)}$$

$$= \frac{1}{(\sqrt{3}+1)} = \frac{(\sqrt{3}-1)}{2}$$

90. We have

$$\frac{R}{r} = \frac{\frac{abc}{4\Delta}}{\frac{\Delta}{s}} = \frac{a^3}{4\Delta} \times \frac{s}{\Delta} = \frac{a^3}{4\Delta} \times \frac{3a}{2\Delta}$$

$$= \frac{3a^4}{8} \times \frac{1}{\Delta^2}$$

$$= \frac{3a^4}{8} \times \frac{1}{\frac{3a^4}{16}} = \frac{3a^4}{8} \times \frac{16}{3a^4} = 2$$

91. We have

$$\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca} = \frac{a+b+c}{abc}$$

$$= \frac{2s}{4\Delta R}$$

$$= \frac{\Delta}{2\Delta R}$$

$$= \frac{1}{2rR}$$

92. We have

$$\cos A + \cos B + \cos C$$

$$= 2 \cos\left(\frac{A+B}{2}\right) \cos\left(\frac{A-B}{2}\right) + \cos C$$

$$= 2 \cos\left(\frac{\pi-C}{2}\right) \cos\left(\frac{A-B}{2}\right) + \cos C$$

$$= 2 \sin\left(\frac{C}{2}\right) \cos\left(\frac{A-B}{2}\right) + \cos C$$

$$= 2 \sin\left(\frac{C}{2}\right) \cos\left(\frac{A-B}{2}\right) + 1 - 2 \sin^2\left(\frac{C}{2}\right)$$

$$= 1 + 2 \sin\left(\frac{C}{2}\right) \left(\cos\left(\frac{A-B}{2}\right) - \sin\left(\frac{C}{2}\right) \right)$$

$$= 1 + 2 \sin\left(\frac{C}{2}\right) \left(\cos\left(\frac{A-B}{2}\right) - \cos\left(\frac{A+B}{2}\right) \right)$$

$$= 1 + 2 \sin\left(\frac{C}{2}\right) \left(2 \sin\left(\frac{A}{2}\right) \sin\left(\frac{B}{2}\right) \right)$$

$$= 1 + 4 \sin\left(\frac{A}{2}\right) \sin\left(\frac{B}{2}\right) \sin\left(\frac{C}{2}\right)$$

$$= \left(1 + \frac{r}{R}\right)$$

93. We have,

$$\begin{aligned} \sin A + \sin B + \sin C &= \frac{a}{2R} + \frac{b}{2R} + \frac{c}{2R} \\ &= \frac{a+b+c}{2R} \\ &= \frac{2s}{2R} \\ &= \frac{s}{R} \\ &= \frac{\Delta}{rR} \end{aligned}$$

94. We have

$$\begin{aligned} a \cot A + b \cot B + c \cot C &= \left[2R \sin A \times \frac{\cos A}{\sin A} + 2R \sin B \times \frac{\cos B}{\sin B} \right. \\ &\quad \left. + 2R \sin C \times \frac{\cos C}{\sin C} \right] \\ &= 2R(\cos A + \cos B + \cos C) \\ &= 2R \left(1 + \frac{r}{R} \right) \\ &= 2(R+r) \end{aligned}$$

95. We have $r_1 = R$

$$\begin{aligned} 4R \sin\left(\frac{A}{2}\right) \cos\left(\frac{B}{2}\right) \cos\left(\frac{C}{2}\right) &= R \\ 2 \sin\left(\frac{A}{2}\right) \left\{ 2 \cos\left(\frac{B}{2}\right) \cos\left(\frac{C}{2}\right) \right\} &= 1 \\ 2 \sin\left(\frac{A}{2}\right) \left\{ \cos\left(\frac{B+C}{2}\right) + \cos\left(\frac{B-C}{2}\right) \right\} &= 1 \\ 2 \cos\left(\frac{B+C}{2}\right) \left\{ \cos\left(\frac{B+C}{2}\right) + \cos\left(\frac{B-C}{2}\right) \right\} &= 1 \\ 2 \cos^2\left(\frac{B+C}{2}\right) + \left\{ 2 \cos\left(\frac{B+C}{2}\right) \cos\left(\frac{B-C}{2}\right) \right\} &= 1 \end{aligned}$$

$$\begin{aligned} (1 + \cos(B+C)) &= \cos(B) + \cos(C) = 1 \\ \cos(B) + \cos(C) &= -\cos(B+C) = \cos A \end{aligned}$$

Hence, the result.

96. We have

$$\begin{aligned} (b+c) \tan\left(\frac{A}{2}\right) &= 2R(\sin B + \sin C) \tan\left(\frac{A}{2}\right) \\ &= 2R \times 2 \sin\left(\frac{B+C}{2}\right) \cos\left(\frac{B-C}{2}\right) \times \tan\left(\frac{A}{2}\right) \\ &= 2R \times 2 \cos\left(\frac{A}{2}\right) \cos\left(\frac{B-C}{2}\right) \times \frac{\sin\left(\frac{A}{2}\right)}{\cos\left(\frac{A}{2}\right)} \end{aligned}$$

$$\begin{aligned} &= 2R \times 2 \cos\left(\frac{B-C}{2}\right) \times \sin\left(\frac{A}{2}\right) \\ &= 2R \times 2 \cos\left(\frac{B-C}{2}\right) \times \cos\left(\frac{B+C}{2}\right) \\ &= 2R \times (\cos(B) + \cos(C)) \end{aligned}$$

Thus, LHS

$$\begin{aligned} &= 4R(\cos A + \cos B + \cos C) \\ &= 4R \left(1 + \frac{r}{R} \right) \\ &= 4(R+r) \end{aligned}$$

97. We have

$$\begin{aligned} R &= \frac{c}{2 \sin C} = \frac{c}{2 \sin(90^\circ)} = \frac{c}{2} \\ c &= 2R \end{aligned}$$

$$\text{Also, } r = (s-c) \tan\left(\frac{C}{2}\right)$$

$$\begin{aligned} r &= (s-c) \tan(45^\circ) = (s-c) \\ 2r &= (2s-2c) = (a+b+c-2c) \\ 2r &= (a+b-c) \\ 2r &= (a+b-2R) \\ 2(R+r) &= (a+b) \end{aligned}$$

98. We have

$$\begin{aligned} \cos^2 \frac{A}{2} + \cos^2 \frac{B}{2} + \cos^2 \frac{C}{2} &= \frac{1}{2} \left(2 \cos^2 \left(\frac{A}{2} \right) + 2 \cos^2 \left(\frac{B}{2} \right) + 2 \cos^2 \left(\frac{C}{2} \right) \right) \\ &= \frac{1}{2} (1 + \cos(A) + 1 + \cos(B) + 1 + \cos(C)) \\ &= \frac{1}{2} (3 + \cos(A) + \cos(B) + \cos(C)) \\ &= \frac{1}{2} \left(3 + \left(1 + \frac{r}{R} \right) \right) \\ &= \frac{1}{2} \left(4 + \frac{r}{R} \right) \\ &= \left(2 + \frac{r}{2R} \right) \end{aligned}$$

99. Let O is the circum-centre and $OD = x$, $OE = y$, $OF = z$ respectively.

$$\text{Also, } OA = R = OB = OC$$

$$\text{We have } x = OD = R \cos A$$

$$\Rightarrow \frac{a}{2 \sin A} \cdot \cos A = \frac{a}{2 \tan A}$$

$$\tan A = \frac{a}{2x}$$

$$\text{Similarly, } \tan B = \frac{b}{2y} \text{ and } \tan C = \frac{c}{2z}$$

As we know that, in a $\triangle ABC$,

$$\tan A + \tan B + \tan C = \tan A \cdot \tan B \cdot \tan C$$

$$\Rightarrow \frac{a}{2x} + \frac{b}{2y} + \frac{c}{2z} = \frac{a}{2x} \cdot \frac{b}{2y} \cdot \frac{c}{2z}$$

$$\Rightarrow \frac{a}{x} + \frac{b}{y} + \frac{c}{z} = \frac{a \cdot b \cdot c}{4 \cdot x \cdot y \cdot z}$$

100. We have

$$R_1 = \frac{OB \cdot OC \cdot BC}{4\Delta_{OBC}} = \frac{R \cdot R \cdot a}{4\Delta_1} = \frac{R^2 \cdot a}{4\Delta_1}$$

$$\Rightarrow \frac{a}{R_1} = \frac{4\Delta_1}{R^2}$$

Similarly, $\frac{b}{R_2} = \frac{4\Delta_2}{R^2}$ and $\frac{c}{R_3} = \frac{4\Delta_3}{R^2}$

Thus,

$$\begin{aligned} \frac{a}{R_1} + \frac{b}{R_2} + \frac{c}{R_3} &= \frac{4\Delta_1}{R^2} + \frac{4\Delta_2}{R^2} + \frac{4\Delta_3}{R^2} \\ &= \frac{4(\Delta_1 + \Delta_2 + \Delta_3)}{R^2} \\ &= \frac{4\Delta}{R^2} \\ &= \frac{4\Delta}{R^2} \end{aligned}$$

101. Let $AD = p_1$, $BE = p_2$ and $CF = p_3$

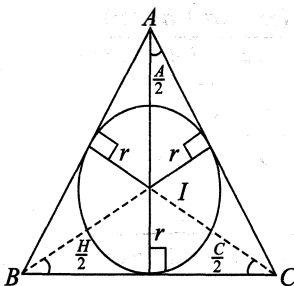
Then, $\Delta = \frac{1}{2}ap_1 = \frac{1}{2}bp_2 = \frac{1}{2}cp_3$

$$\Rightarrow p_1 = \frac{2\Delta}{a}, p_2 = \frac{2\Delta}{b}, p_3 = \frac{2\Delta}{c}$$

We have $p_1 \cdot p_2 \cdot p_3 = \frac{2\Delta}{a} \cdot \frac{2\Delta}{b} \cdot \frac{2\Delta}{c}$

$$\begin{aligned} &= \frac{8\Delta^3}{abc} \\ &= \frac{8 \left(\frac{abc}{4R} \right)^3}{abc} \end{aligned}$$

102.



Since IA is the internal angle bisector of $\angle A$, so we can write

$$\begin{aligned} \frac{AB}{AC} &= \frac{BD}{DC} \\ \Rightarrow \frac{BD}{DC} &= \frac{AB}{AC} = \frac{c}{b} \end{aligned}$$

$$\Rightarrow \frac{DC}{BD} + 1 = \frac{b}{c} + 1$$

$$\Rightarrow \frac{BC}{BD} = \frac{b+c}{c}$$

$$\Rightarrow BD = \frac{ac}{b+c}$$

In ΔABD , $\frac{BD}{\sin\left(\frac{A}{2}\right)} = \frac{AD}{\sin B}$

$$\begin{aligned} \Rightarrow AD &= \frac{BD \sin B}{\sin\left(\frac{A}{2}\right)} \\ &= \frac{ac \sin B}{(b+c) \sin\left(\frac{A}{2}\right)} = \frac{2\Delta}{(b+c) \sin(A/2)} \end{aligned}$$

similarly, $BE = \frac{2\Delta}{(c+a) \sin\left(\frac{B}{2}\right)}$

and $CF = \frac{2\Delta}{(a+b) \sin\left(\frac{C}{2}\right)}$

103. Now $2s = a + b + c = 18 + 24 + 30 = 72$

$$\Rightarrow 2s = 72$$

$$\Rightarrow s = 36$$

We have $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$

$$\begin{aligned} &= \sqrt{36(36-18)(36-24)(36-30)} \\ &= \sqrt{36 \times 18 \times 12 \times 6} \\ &= \sqrt{36 \times 9 \times 12 \times 12} \\ &= 6 \times 3 \times 12 \\ &= 216 \end{aligned}$$

Thus, $r_1 = \frac{\Delta}{s-a} = \frac{216}{36-18} = \frac{216}{18} = 12$

$$r_2 = \frac{\Delta}{s-b} = \frac{216}{36-24} = \frac{216}{12} = 18$$

and $r_3 = \frac{\Delta}{s-c} = \frac{216}{36-30} = \frac{216}{6} = 36$

104. $\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = \frac{s-a}{\Delta} + \frac{s-b}{\Delta} + \frac{s-c}{\Delta}$

$$\begin{aligned} &= \frac{3s - (a+b+c)}{\Delta} \\ &= \frac{3s - 2s}{\Delta} \\ &= \frac{s}{\Delta} \\ &= \frac{1}{r} \end{aligned}$$

$$\begin{aligned}
 105. \quad & \frac{b-c}{r_1} + \frac{c-a}{r_2} + \frac{a-b}{r_3} \\
 &= \frac{(b-c)(s-a)}{\Delta} + \frac{(c-a)(s-b)}{\Delta} + \frac{(a-b)(s-c)}{\Delta} \\
 &= \frac{1}{\Delta} [s(b-c+c-a+a-b)] \\
 &\quad - \frac{1}{\Delta} (ab-ac+bc-ab+ca-bc) \\
 &= 0-0 \\
 &= 0
 \end{aligned}$$

$$\begin{aligned}
 106. \quad & \text{Given, } \frac{s-c}{s-a} = \frac{b-c}{a-b} \\
 \Rightarrow & \frac{2s-2c}{2s-2a} = \frac{b-c}{a-b} \\
 \Rightarrow & \frac{a+b-c}{b+c-a} = \frac{b-c}{a-b} \\
 \Rightarrow & \frac{a+b-c}{b-c} = \frac{b+c-a}{a-b} \\
 \Rightarrow & \frac{a}{b-c} + 1 = \frac{c}{a-b} - 1 \\
 \Rightarrow & \frac{c}{a-b} + \frac{a}{b-c} = 2 \\
 \Rightarrow & \frac{c(b-c) + a(a-b)}{(a-b)(b-c)} = 2 \\
 \Rightarrow & \frac{bc - c^2 + a^2 - ab}{(a-b)(b-c)} = 2 \\
 \Rightarrow & bc - c^2 + a^2 - ab = 2(ab - ac - b^2 + bc) \\
 \Rightarrow & 2b^2 - b - c^2 + a^2 - 3ab + 2ac = 0
 \end{aligned}$$

107. We have

$$\begin{aligned}
 & \left(1 - \frac{r_1}{r_2}\right) \left(1 - \frac{r_1}{r_3}\right) = 2 \\
 & \left(1 - \frac{s-b}{s-a}\right) \left(1 - \frac{s-c}{s-a}\right) = 2 \\
 & \left(\frac{s-a-s+b}{s-a}\right) \left(\frac{s-a-s+c}{s-a}\right) = 2 \\
 & \left(\frac{a-b}{s-a}\right) \left(\frac{a-c}{s-a}\right) = 2 \\
 & \frac{(a-b)(a-c)}{(s-a)^2} = 2 \\
 & (a-b)(a-c) = 2(s-a)^2 \\
 & 2(a-b)(a-c) = 4(s-a)^2 = (2s-2a)^2 \\
 & 2(a-b)(a-c) = (b+c-a)^2 \\
 & 2(a^2 - ab - ac + bc) \\
 & \quad = a^2 + b^2 + c^2 + 2bc - 2ab - 2ac \\
 & a^2 + b^2 = c^2
 \end{aligned}$$

Thus, the ΔABC is a right angled.

108. We have

$$\begin{aligned}
 & r_1 + r_2 + r_3 - r \\
 &= \left\{ \frac{\Delta}{(s-a)} + \frac{\Delta}{(s-b)} \right\} + \left\{ \frac{\Delta}{(s-c)} - \frac{\Delta}{s} \right\} \\
 &= \left\{ \frac{\Delta(s-b+s-a)}{(s-a)(s-b)} \right\} + \left\{ \frac{\Delta(s-s+c)}{s(s-c)} \right\} \\
 &= \left\{ \frac{\Delta(2s-b-a)}{(s-a)(s-b)} \right\} + \left\{ \frac{\Delta(c)}{s(s-c)} \right\} \\
 &= \left\{ \frac{\Delta(a+b+c-b-a)}{(s-a)(s-b)} \right\} + \left\{ \frac{\Delta(c)}{s(s-c)} \right\} \\
 &= \left\{ \frac{\Delta(c)}{(s-a)(s-b)} \right\} + \left\{ \frac{\Delta(c)}{s(s-c)} \right\} \\
 &= \Delta(c) \left\{ \frac{1}{(s-a)(s-b)} + \frac{1}{s(s-c)} \right\} \\
 &= \Delta(c) \left\{ \frac{s(s-c) + (s-a)(s-b)}{s(s-a)(s-b)(s-c)} \right\} \\
 &= \Delta(c) \left\{ \frac{s^2 - cs + s^2 - (a+b)s + ab}{\Delta^2} \right\} \\
 &= c \times \left\{ \frac{2s^2 - (a+b+c)s + ab}{\Delta} \right\} \\
 &= c \times \left\{ \frac{2s^2 - 2s^2 + ab}{\Delta} \right\} \\
 &= \frac{abc}{\Delta} = 4R
 \end{aligned}$$

109. $r_1 r_2 + r_2 r_3 + r_3 r_1$

$$\begin{aligned}
 &= \frac{\Delta}{(s-a)} \cdot \frac{\Delta}{(s-b)} + \frac{\Delta}{(s-b)} \\
 &\quad \cdot \frac{\Delta}{(s-c)} + \frac{\Delta}{(s-c)} \cdot \frac{\Delta}{(s-a)} \\
 &= \frac{\Delta^2(s-c+s-a+s-b)}{(s-a)(s-b)(s-c)} \\
 &= \frac{\Delta^2(3s-(a+b+c))}{(s-a)(s-b)(s-c)} \\
 &= \frac{\Delta^2(3s-2s)}{(s-a)(s-b)(s-c)} \\
 &= \frac{\Delta^2 \times s}{(s-a)(s-b)(s-c)} \\
 &= \frac{\Delta^2 \times s^2}{s(s-a)(s-b)(s-c)} \\
 &= \frac{\Delta^2 \times s^2}{\Delta^2} \\
 &= s^2
 \end{aligned}$$

$$\begin{aligned}
 110. \quad r_1 + r_2 - r_3 + r &= (r_1 + r_2) - (r_3 - r) \\
 &= \left(\frac{\Delta}{s-a} + \frac{\Delta}{s-b} \right) - \left(\frac{\Delta}{s-c} - \frac{\Delta}{s} \right) \\
 &= \Delta \left\{ \left(\frac{1}{s-a} + \frac{1}{s-b} \right) - \left(\frac{1}{s-c} - \frac{1}{s} \right) \right\} \\
 &= \Delta \left\{ \left(\frac{(s-a+s-b)}{(s-a)(s-b)} \right) - \left(\frac{s-(s-c)}{s(s-c)} \right) \right\} \\
 &= \Delta \left\{ \left(\frac{(2s-(a+b))}{(s-a)(s-b)} \right) - \left(\frac{c}{s(s-c)} \right) \right\} \\
 &= \Delta \left\{ \left(\frac{((a+b+c)-(a+b))}{(s-a)(s-b)} \right) - \left(\frac{c}{s(s-c)} \right) \right\} \\
 &= \Delta c \left\{ \left(\frac{1}{(s-a)(s-b)} - \frac{1}{s(s-c)} \right) \right\} \\
 &= \Delta c \left\{ \left(\frac{s(s-c) - (s-a)(s-b)}{(s-a)(s-b)s(s-c)} \right) \right\} \\
 &= \Delta c \left\{ \left(\frac{s^2 - cs - (s^2 - (a+b)s + ab)}{s(s-a)(s-b)(s-c)} \right) \right\} \\
 &= \frac{c}{\Delta} \times ((a+b-c)s - ab) \\
 &= \frac{c}{2\Delta} \times ((a+b-c)2s - 2ab) \\
 &= \frac{c}{2\Delta} \times ((a+b-c)(a+b+c) - 2ab) \\
 &= \frac{c}{2\Delta} \times ((a+b)^2 - c^2 - 2ab) \\
 &= \frac{c}{2\Delta} \times (a^2 + b^2 - c^2) \\
 &= \frac{2abc}{2\Delta} \times \left(\frac{a^2 + b^2 - c^2}{2ab} \right) \\
 &= \frac{abc}{\Delta} \times \cos(C) \\
 &= 4R \cos(C)
 \end{aligned}$$

111. Given, r_1, r_2, r_3 are in HP

$$\begin{aligned}
 \Rightarrow r_2 &= \frac{2r_1 r_3}{r_1 + r_3} \\
 \Rightarrow \frac{\Delta}{s-b} &= \frac{2 \cdot \frac{\Delta}{s-a} \cdot \frac{\Delta}{s-c}}{\frac{\Delta}{s-a} + \frac{\Delta}{s-c}} \\
 \Rightarrow \frac{1}{s-b} &= \frac{2 \cdot \frac{1}{s-a} \cdot \frac{1}{s-c}}{\frac{1}{s-a} + \frac{1}{s-c}}
 \end{aligned}$$

$$\begin{aligned}
 \Rightarrow \frac{1}{s-b} \left(\frac{1}{s-a} + \frac{1}{s-c} \right) &= 2 \cdot \frac{1}{s-a} \cdot \frac{1}{s-c} \\
 \Rightarrow \frac{1}{s-b} \left(\frac{s-c+s-a}{(s-a)(s-c)} \right) &= \frac{2}{(s-a)(s-c)} \\
 \Rightarrow (2s-a-c) &= 2(s-b) \\
 \Rightarrow a+c &= 2b \\
 \Rightarrow a, b, c &\in \text{AP}
 \end{aligned}$$

112. Since a, b, c are in AP as well as in GP, so $a = b = c$

$$\text{Now, } r_1 = \frac{\Delta}{s-a} = r_2 = r_3$$

$$\begin{aligned}
 \text{Thus, } \left(\frac{r_1}{r_2} - \frac{r_2}{r_3} + 10 \right) &= 1 - 1 + 10 = 10
 \end{aligned}$$

113. We have

$$\begin{aligned}
 \frac{1}{r_1^2} + \frac{1}{r_2^2} + \frac{1}{r_3^2} + \frac{1}{r^2} &= \frac{(s-a)^2}{\Delta^2} + \frac{(s-b)^2}{\Delta^2} + \frac{(s-c)^2}{\Delta^2} + \frac{s^2}{\Delta^2} \\
 &= \frac{1}{\Delta^2} [4s^2 - 2(a+b+c)s + (a^2 + b^2 + c^2)] \\
 &= \frac{1}{\Delta^2} [4s^2 - 2 \cdot 2s \cdot s + (a^2 + b^2 + c^2)] \\
 &= \frac{1}{\Delta^2} [4s^2 - 4s^2 + (a^2 + b^2 + c^2)] \\
 &= \frac{(a^2 + b^2 + c^2)}{\Delta^2}
 \end{aligned}$$

114. We have

$$\begin{aligned}
 (r_1 - r)(r_2 - r)(r_3 - r) &= \left(\frac{\Delta}{s-a} - \frac{\Delta}{s} \right) \left(\frac{\Delta}{s-b} - \frac{\Delta}{s} \right) \left(\frac{\Delta}{s-c} - \frac{\Delta}{s} \right) \\
 &= \Delta^3 \left(\frac{1}{s-a} - \frac{1}{s} \right) \left(\frac{1}{s-b} - \frac{1}{s} \right) \left(\frac{1}{s-c} - \frac{1}{s} \right) \\
 &= \Delta^3 \left(\frac{s-s+a}{s(s-a)} \right) \left(\frac{s-s+b}{s(s-b)} \right) \left(\frac{s-s+c}{s(s-c)} \right) \\
 &= \frac{\Delta^3}{s^2} \left(\frac{abc}{(s-a)(s-b)(s-c)} \right) \\
 &= \frac{\Delta^3}{s^2} \left(\frac{abc}{\Delta^2} \right) \\
 &= \frac{\Delta \times abc}{s^2} \\
 &= 4 \times \left(\frac{\Delta}{s} \right)^2 \times \left(\frac{abc}{4\Delta} \right) \\
 &= 4r^2 R
 \end{aligned}$$

115. We have

$$r_1 = \frac{\Delta}{s-a} = 1, r_2 = \frac{\Delta}{s-b} = 2$$

and $r_3 = \frac{\Delta}{s-c}$

$$\Rightarrow s-a = \Delta, s-b = \frac{\Delta}{2}$$

and $s-c = \frac{\Delta}{r_3}$

$$\Rightarrow c = \Delta \left(1 + \frac{1}{2}\right), a = \Delta \left(\frac{1}{2} + \frac{1}{r_3}\right),$$

$$b = \Delta \left(1 + \frac{1}{r_3}\right)$$

Since triangle is right angled, so

$$a^2 + b^2 = c^2$$

$$\Rightarrow \Delta^2 \left(\frac{3}{2}\right)^2 = \frac{\Delta^2 (r_3 + 2)^2}{4(r_3)^2} + \Delta^2 \left(\frac{r_3 + 1}{r_3}\right)^2$$

$$\Rightarrow \left(\frac{3}{2}\right)^2 = \frac{(r_3 + 2)^2}{4(r_3)^2} + \left(\frac{r_3 + 1}{r_3}\right)^2$$

$$\Rightarrow 9r_3^2 = (r_3 + 2)^2 + 4(r_3 + 1)^2$$

$$\Rightarrow 4r_3^2 - 12r_3 - 8 = 0$$

$$\Rightarrow r_3^2 - 3r_3 - 2 = 0$$

$$\Rightarrow r_3 = \frac{3 \pm \sqrt{17}}{2} = \frac{3 + \sqrt{17}}{2}$$

as r_3 is positive.

116. Let a, b be the sides of a triangle.

Then $a + b = 5$ and $ab = 3$

Now, $\cos C = \frac{a^2 + b^2 - c^2}{2ab}$

$$\Rightarrow \cos\left(\frac{\pi}{3}\right) = \frac{19 - c^2}{6}$$

$$\Rightarrow \frac{1}{2} = \frac{19 - c^2}{6}$$

$$\Rightarrow 19 - c^2 = 3$$

$$\Rightarrow c = 4$$

Thus, $r \cdot R = \frac{\Delta}{s} \times \frac{abc}{4\Delta}$

$$= \frac{abc}{4s} = \frac{abc}{2(a+b+c)} = \frac{3 \cdot 4}{2(5+4)} = \frac{12}{18} = \frac{2}{3}$$

117. By sine rule, $\frac{a}{\sin(120^\circ)} = \frac{b}{\sin(30^\circ)}$

$$\Rightarrow \frac{a}{\sqrt{3}/2} = \frac{b}{1/2}$$

$$\Rightarrow a = b\sqrt{3}$$

Also, from the above figure, $r = \sqrt{3}$

$$\Rightarrow \frac{\sqrt{3}}{a/2} = \tan(15^\circ)$$

$$\Rightarrow \frac{2\sqrt{3}}{a} = (2 - \sqrt{3})$$

$$\Rightarrow a = \frac{2\sqrt{3}}{(2 - \sqrt{3})}$$

Now, $b = \frac{a}{\sqrt{3}} = \frac{2\sqrt{3}}{\sqrt{3}(2 - \sqrt{3})} \times \frac{1}{\sqrt{3}} = \frac{2}{(2 - \sqrt{3})}$

Thus, the required area

$$= \frac{1}{2} \times ab \times \sin(30^\circ)$$

$$= \frac{1}{2} \times \frac{2\sqrt{3}}{(2 - \sqrt{3})} \times \frac{2}{(2 - \sqrt{3})} \times \frac{1}{2}$$

$$= \sqrt{3} \times (2 + \sqrt{3})^2$$

$$= \sqrt{3} \times (7 + 4\sqrt{3})$$

$$= (12 + 7\sqrt{3}) \text{ sq. u.}$$

118. Given, $r = r_1 - r_2 - r_3$

$$\Rightarrow r_1 - r = r_2 + r_3$$

$$\Rightarrow \frac{\Delta}{s-a} - \frac{\Delta}{s} = \frac{\Delta}{s-b} + \frac{\Delta}{s-c}$$

$$\Rightarrow \frac{1}{s-a} - \frac{1}{s} = \frac{1}{s-b} + \frac{1}{s-c}$$

$$\Rightarrow \frac{(s-a-s)}{(s-a)} = \frac{(s-c+s-b)}{(s-b)(s-c)}$$

$$\Rightarrow \frac{a}{(s-a)} = \frac{a}{(s-b)(s-c)}$$

$$\Rightarrow \frac{(s-b)(s-c)}{(s-a)} = 1$$

$$\Rightarrow \tan^2\left(\frac{A}{2}\right) = 1$$

$$\Rightarrow \tan\left(\frac{A}{2}\right) = 1$$

$$\Rightarrow \tan\left(\frac{A}{2}\right) = \tan\left(\frac{\pi}{4}\right)$$

$$\Rightarrow A = \frac{\pi}{2}$$

Thus, the triangle is right angled.

119. We have $r \cdot r_1 \cdot r_2 \cdot r_3$

$$= \frac{\Delta}{s} \cdot \frac{\Delta}{(s-a)} \cdot \frac{\Delta}{(s-b)} \cdot \frac{\Delta}{(s-c)}$$

$$= \frac{\Delta^4}{s(s-a)(s-b)(s-c)}$$

$$= \frac{\Delta^4}{\Delta^2}$$

$$= \Delta^2$$

120. We have

$$\begin{aligned} \frac{r_1 + r_2}{1 + \cos C} &= \frac{\Delta}{s-a} + \frac{\Delta}{s-b} \\ &= \frac{2 \cos^2 \left(\frac{C}{2} \right)}{2 \cos^2 \left(\frac{C}{2} \right)} \\ &= \frac{\Delta(s-a+s-b)}{(s-a)(s-b)} \\ &= \frac{\Delta(2s-a-b)}{2 \cos^2 \left(\frac{C}{2} \right)} \\ &= \frac{\Delta(2s-a-b)}{(s-a)(s-b) \times 2 \times \frac{s(s-c)}{ab}} \\ &= \frac{\Delta \times abc}{s(s-a)(s-b)(s-c)} \\ &= \frac{\Delta \times abc}{\Delta^2} = \frac{abc}{\Delta} \end{aligned}$$

Similarly, we can easily prove that,

$$\frac{(r_2 + r_3)}{1 + \cos A} = \frac{abc}{\Delta} \text{ and } \frac{(r_3 + r_1)}{1 + \cos B} = \frac{abc}{\Delta}$$

Thus, $\frac{(r_2 + r_3)}{1 + \cos A} = \frac{abc}{\Delta}$ and $\frac{(r_3 + r_1)}{1 + \cos B} = \frac{abc}{\Delta}$

121. We have

$$\begin{aligned} &\left(\frac{1}{r} - \frac{1}{r_1} \right) \left(\frac{1}{r} - \frac{1}{r_2} \right) \left(\frac{1}{r} - \frac{1}{r_3} \right) \\ &= \frac{1}{\Delta^3} (s-s+a)(s-s+b)(s-s+c) \\ &= \frac{1}{\Delta^3} \times abc \\ &= 16 \times \frac{abc}{4\Delta} \times \frac{s^2}{\Delta^2 (2s)^2} \\ &= \frac{16R}{r^2 (a+b+c)^2} \end{aligned}$$

122. We have

$$\begin{aligned} &\left(\frac{1}{r_1} + \frac{1}{r_2} \right) \left(\frac{1}{r_2} + \frac{1}{r_3} \right) \left(\frac{1}{r_3} + \frac{1}{r_1} \right) \\ &= \left(\frac{s-a}{\Delta} + \frac{s-b}{\Delta} \right) \left(\frac{s-b}{\Delta} + \frac{s-c}{\Delta} \right) \left(\frac{s-c}{\Delta} + \frac{s-a}{\Delta} \right) \\ &= \frac{1}{\Delta^3} \times abc \\ &= \frac{1}{\left(\frac{abc}{4R} \right)^3} \times abc \\ &= \frac{64R^3}{(abc)^2} \end{aligned}$$

123. We have

$$\begin{aligned} (r_1 + r_2 + r_3 - r)^2 &= r_1^2 + r_2^2 + r_3^2 + r^2 - 2r(r_1 + r_2 + r_3) \\ &\quad + 2(r_1 r_2 + r_2 r_3 + r_3 r_1) \end{aligned}$$

Now, $(r_1 + r_2 + r_3 - r) = 4R$

$$(r_1 r_2 + r_2 r_3 + r_3 r_1) = s^2$$

and $2r(r_1 + r_2 + r_3)$

$$\begin{aligned} &= 2 \times \frac{\Delta}{s} \left(\frac{\Delta}{(s-a)} + \frac{\Delta}{(s-a)} + \frac{\Delta}{(s-a)} \right) \\ &= 2 \times \frac{\Delta^2}{s} \left(\frac{1}{(s-a)} + \frac{1}{(s-b)} + \frac{1}{(s-c)} \right) \\ &= 2 \times \frac{\Delta^2}{s} \left(\frac{(s-b)(s-c) + (s-a)(s-c) + (s-a)(s-b)}{(s-a)(s-b)(s-c)} \right) \\ &= 2 \times \Delta^2 \left(\frac{3s^2 - 2(a+b+c)s + (ab+bc+ca)}{s(s-a)(s-b)(s-c)} \right) \\ &= 2 \times \Delta^2 \left(\frac{3s^2 - 2 \cdot 2s \cdot s + (ab+bc+ca)}{\Delta^2} \right) \end{aligned}$$

$$= 2 \times ((ab + bc + ca) - s^2)$$

Thus, $(r_1^2 + r_2^2 + r_3^2 + r^2)$

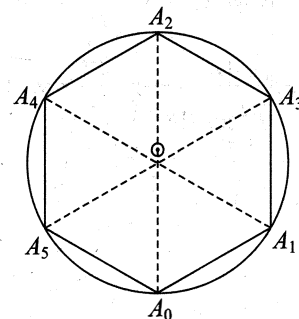
$$\begin{aligned} &= (r_1 + r_2 + r_3 - r)^2 + 2r(r_1 + r_2 + r_3) \\ &\quad - 2(r_1 r_2 + r_2 r_3 + r_3 r_1) \\ &= 16R^2 + 2(ab + bc + ca - s^2) - 2s^2 \\ &= 16R^2 + 2(ab + bc + ca) - (2s)^2 \\ &= 16R^2 + 2(ab + bc + ca) - (a+b+c)^2 \\ &= 16R^2 - (a^2 + b^2 + c^2) \end{aligned}$$

Hence, the result.

124. In a triangle ΔABC , prove that

$$\frac{(r_1 + r_2)(r_2 + r_3)(r_3 + r_1)}{(r_1 r_2 + r_2 r_3 + r_3 r_1)} = 4R$$

125.



Here, $OA_0 = OA_1 = OA_2 = \dots = OA_5 = 1$.

and

$$\angle A_0 O A_1 = \frac{2\pi}{6} = \angle A_1 O A_2 = \dots = \angle A_4 O A_5$$

Now, $\cos \left(\frac{\pi}{3} \right) = \frac{OA_0^2 + OA_1^2 - A_1 A_2}{2 OA_0 \cdot OA_1}$

$$\Rightarrow \frac{1}{2} = \frac{1+1 - A_1 A_2}{2 \cdot 1 \cdot 1}$$

$$\Rightarrow \frac{1}{2} = \frac{2 - A_0 A_1^2}{2}$$

$$\Rightarrow A_0 A_1^2 = 1$$

$$\Rightarrow A_0 A_1 = 1$$

$$\text{Also, } \cos\left(\frac{2\pi}{3}\right) = \frac{OA_0^2 + OA_1^2 - A_0 A_2^2}{2 \cdot OA_0 \cdot OA_1}$$

$$\Rightarrow \frac{1}{2} = \frac{1 + 1 - A_0 A_2^2}{2 \cdot 1 \cdot 1}$$

$$\Rightarrow \frac{1}{2} = \frac{2 - A_0 A_2^2}{2}$$

$$\Rightarrow A_0 A_2 = \sqrt{3}$$

$$\text{Again, } \cos\left(\frac{4\pi}{3}\right) = \frac{OA_0^2 + OA_4^2 - A_0 A_4^2}{2 \cdot OA_0 \cdot OA_4}$$

$$= \frac{1 + 1 - A_0 A_4^2}{2 \cdot 1 \cdot 1}$$

$$= \frac{2 - A_0 A_4^2}{2}$$

$$\Rightarrow \frac{1}{2} = \frac{2 - A_0 A_4^2}{2}$$

$$\Rightarrow A_0 A_4 = \sqrt{3}$$

Hence, the value of

$$\Rightarrow A_0 A_1 \cdot A_0 A_2 \cdot A_0 A_4 = 1 \cdot \sqrt{3} \cdot \sqrt{3} = 3$$

126. Let R be the radius of the circle.

$$\text{Then, } A_1 = \pi R^2$$

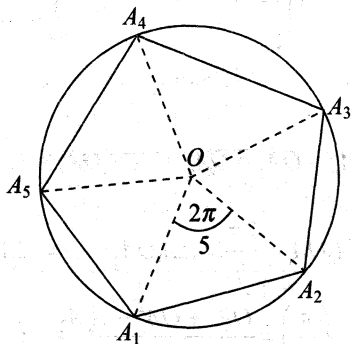
$$\text{and } \Delta = \frac{nR^2}{2} \sin\left(\frac{2\pi}{n}\right)$$

$$A_2 = \frac{5 \cdot R^2}{2} \sin\left(\frac{360^\circ}{5}\right)$$

$$= \frac{5}{2} R^2 \sin(72^\circ) = \frac{5R^2}{2} \times \cos(18^\circ)$$

$$\text{Now, } \frac{A_1}{A_2} = \frac{\pi R^2}{\frac{5}{2} R^2 \cos(18^\circ)} = \frac{2\pi}{5} \times \sec(18^\circ)$$

127.



$$\text{Here, } OA_1 = OA_2 = OA_3 = OA_4 = OA_5 = 1$$

and

$$\angle A_1 O A_2 = \frac{2\pi}{5} = \angle A_2 O A_3 = \dots = \angle A_4 O A_5$$

$$\text{Now, } \cos\left(\frac{2\pi}{5}\right) = \frac{OA_1^2 + OA_2^2 - A_1 A_2^2}{2 \cdot OA_1 \cdot OA_2}$$

$$\Rightarrow \sin(18^\circ) = \frac{1 + 1 - A_1 A_2^2}{2 \cdot 1 \cdot 1}$$

$$\Rightarrow \frac{\sqrt{5} - 1}{4} = \frac{2 - A_1 A_2^2}{2}$$

$$\Rightarrow A_1 A_2^2 = 2 - \frac{\sqrt{5} - 1}{2} = \frac{5 - \sqrt{5}}{2}$$

$$\Rightarrow A_1 A_2 = \sqrt{\frac{5 - \sqrt{5}}{2}}$$

$$\text{Similarly, } A_1 A_3 = \sqrt{\frac{5 + \sqrt{5}}{2}}$$

Thus, $A_1 A_2 \times A_1 A_3$

$$= \sqrt{\left(\frac{5 - \sqrt{5}}{2}\right) \times \left(\frac{5 + \sqrt{5}}{2}\right)} = \sqrt{\frac{25 - 5}{4}} = \sqrt{\frac{20}{4}} = \sqrt{5}$$

$$128. R = \frac{a}{2} \operatorname{cosec}\left(\frac{\pi}{n}\right),$$

As we know that, the circum-radius of n sided regular polygon

$$= \frac{a}{2} \operatorname{cosec}\left(\frac{\pi}{n}\right), \text{ where } a = \text{side}$$

and $n =$ number of sides

$$= 6 \times \operatorname{cosec}\left(\frac{\pi}{12}\right)$$

$$= 6 \times \operatorname{cosec}(15^\circ)$$

$$= \frac{6 \times 2\sqrt{2}}{\sqrt{3} - 1}$$

$$= 6\sqrt{2}(\sqrt{3} + 1)$$

129. Let the perimeter of the pentagon and the decagon be $10x$.

Then each side of the pentagon is $2x$ and the decagon is x .

Let $A_1 =$ the area of the pentagon

$$= 5x^2 \cot\left(\frac{\pi}{5}\right)$$

and $A_2 =$ the area of the decagon

$$= \frac{5}{2} x^2 \cot\left(\frac{\pi}{10}\right)$$

$$\text{Now, } \frac{A_1}{A_2} = \frac{5x^2 \cot\left(\frac{\pi}{5}\right)}{\frac{5}{2} x^2 \cot\left(\frac{\pi}{10}\right)} = \frac{2 \cot\left(\frac{\pi}{5}\right)}{\cot\left(\frac{\pi}{10}\right)}$$

$$\begin{aligned}
 &= \frac{2 \cot(36^\circ)}{\cot(18^\circ)} = \frac{2 \cos(36^\circ) \sin(18^\circ)}{\sin(36^\circ) \cos(18^\circ)} \\
 &= \frac{2 \cos(36^\circ) \sin(18^\circ)}{2 \sin(18^\circ) \cos^2(18^\circ)} \\
 &= \frac{2 \cos(36^\circ)}{(1 + \cos(36^\circ))} \\
 &= \frac{2 \left(\frac{\sqrt{5} + 1}{4} \right)}{\left(1 + \frac{\sqrt{5} + 1}{4} \right)} \\
 &= \frac{2(\sqrt{5} + 1)}{(\sqrt{5} + 5)} \\
 &= \frac{2(\sqrt{5} + 1)}{\sqrt{5}(\sqrt{5} + 1)} \\
 &= \frac{2}{\sqrt{5}}
 \end{aligned}$$

130. We have

$$R = \frac{2a}{2} \operatorname{cosec} \left(\frac{\pi}{n} \right) = a \operatorname{cosec} \left(\frac{\pi}{n} \right)$$

$$\text{and } r = \frac{2a}{2} \cot \left(\frac{\pi}{n} \right) = a \cot \left(\frac{\pi}{n} \right)$$

Now,

$$\begin{aligned}
 r + R &= a \cot \left(\frac{\pi}{n} \right) + a \operatorname{cosec} \left(\frac{\pi}{n} \right) \\
 &= a \left(\cot \left(\frac{\pi}{n} \right) + \operatorname{cosec} \left(\frac{\pi}{n} \right) \right) \\
 &= a \left(\frac{1 + \cos \left(\frac{\pi}{n} \right)}{\sin \left(\frac{\pi}{n} \right)} \right) \\
 &= a \left(\frac{2 \cos^2 \left(\frac{\pi}{2n} \right)}{2 \sin \left(\frac{\pi}{2n} \right) \cos \left(\frac{\pi}{2n} \right)} \right) \\
 &= a \cot \left(\frac{\pi}{2n} \right)
 \end{aligned}$$

131. Let A_1 be the area of the regular pentagon and A_2 be the area of the regular decagon.

Therefore, $A_1 = A_2$

$$\Rightarrow \frac{5a^2}{4} \cot \left(\frac{\pi}{5} \right) = \frac{6b^2}{4} \cot \left(\frac{\pi}{6} \right)$$

$$\Rightarrow 5a^2 \cot \left(\frac{\pi}{5} \right) = 6b^2 \cot \left(\frac{\pi}{6} \right)$$

$$\Rightarrow 5a^2 \cot(36^\circ) = 6b^2 \cot(30^\circ)$$

$$\Rightarrow 5a^2 \cot(36^\circ) = 6\sqrt{3}b^2$$

$$\Rightarrow \frac{a^2}{6\sqrt{3}} = \frac{b^2}{5 \cot(36^\circ)} = \lambda$$

Hence the ratio of their perimeters

$$\begin{aligned}
 &= \frac{5a}{6b} \\
 &= \frac{5}{6} \times \frac{\sqrt{\lambda 6\sqrt{3}}}{\sqrt{5\lambda \cot(36^\circ)}} = \sqrt{\frac{5\sqrt{3}}{6} \tan(36^\circ)}
 \end{aligned}$$

132. Let the perimeter of the two polygons are nx and $2nx$ respectively.

Then each side of the polygons are $2x$ and x .

Let A_1 = the area of the polygon of n sides

$$= nx^2 \cot \left(\frac{\pi}{n} \right)$$

and A_2 = the area of the decagon

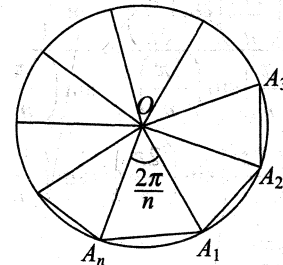
$$= \frac{5}{2} n^2 \cot \left(\frac{\pi}{2n} \right)$$

$$\text{Thus, } \frac{A_1}{A_2} = \frac{5n^2 \cot \left(\frac{\pi}{n} \right)}{\frac{5}{2} n^2 \cot \left(\frac{\pi}{2n} \right)} = \frac{2 \cot \left(\frac{\pi}{n} \right)}{\cot \left(\frac{\pi}{2n} \right)}$$

$$= \frac{2 \cos \left(\frac{\pi}{n} \right) \sin \left(\frac{\pi}{2n} \right)}{2 \sin \left(\frac{\pi}{2n} \right) \cos^2 \left(\frac{\pi}{2n} \right)}$$

$$= \frac{2 \cos \left(\frac{\pi}{n} \right)}{1 + \cos \left(\frac{\pi}{n} \right)}$$

133. Let O be the centre and $A_1 A_2 \dots A_n$ be the regular polygon of n -sides.



Let $OA_1 = OA_2 = \dots = OA_n = r$

and $\angle A_1 O A_2 = \angle A_2 O A_3$

$$= \dots = \angle A_n O A_1 = \frac{2\pi}{n}$$

From the triangle $OA_1 A_2$,

$$\begin{aligned}
 \cos \left(\frac{2\pi}{n} \right) &= \frac{OA_1^2 + OA_2^2 - A_1 A_2^2}{2 \cdot OA_1 \cdot OA_2} \\
 &= \frac{r^2 + r^2 - A_1 A_2^2}{2 \cdot r \cdot r}
 \end{aligned}$$

$$\Rightarrow A_1 A_2 = 2r^2 - 2r^2 \cos\left(\frac{2\pi}{n}\right)$$

$$\Rightarrow A_1 A_2 = 2r^2 \left(1 - \cos\left(\frac{2\pi}{n}\right)\right)$$

$$\begin{aligned} \Rightarrow A_1 A_2 &= 2r^2 \cdot 2 \sin^2\left(\frac{2\pi}{n}\right) \\ &= 4r^2 \cdot \sin^2\left(\frac{2\pi}{n}\right) \end{aligned}$$

$$\Rightarrow A_1 A_2 = 2r \cdot \sin\left(\frac{2\pi}{n}\right)$$

Similarly, $A_1 A_3 = 2r \cdot \sin\left(\frac{4\pi}{n}\right)$

and $A_1 A_4 = 2r \cdot \sin\left(\frac{6\pi}{n}\right)$

Given, $\frac{1}{A_1 A_2} = \frac{1}{A_1 A_3} + \frac{1}{A_1 A_4}$

$$\Rightarrow \frac{1}{2r \cdot \sin\left(\frac{2\pi}{n}\right)} = \frac{1}{2r \cdot \sin\left(\frac{4\pi}{n}\right)} + \frac{1}{2r \cdot \sin\left(\frac{6\pi}{n}\right)}$$

$$\Rightarrow \frac{1}{\sin\left(\frac{\pi}{n}\right)} = \frac{1}{\sin\left(\frac{2\pi}{n}\right)} + \frac{1}{\sin\left(\frac{3\pi}{n}\right)}$$

$$\Rightarrow \frac{1}{\sin\left(\frac{\pi}{n}\right)} - \frac{1}{\sin\left(\frac{3\pi}{n}\right)} = \frac{1}{\sin\left(\frac{2\pi}{n}\right)}$$

$$\Rightarrow \frac{\sin\left(\frac{3\pi}{n}\right) - \sin\left(\frac{\pi}{n}\right)}{\sin\left(\frac{\pi}{n}\right) \sin\left(\frac{3\pi}{n}\right)} = \frac{1}{\sin\left(\frac{2\pi}{n}\right)}$$

$$\Rightarrow \frac{2 \cos\left(\frac{2\pi}{n}\right) \sin\left(\frac{\pi}{n}\right)}{\sin\left(\frac{\pi}{n}\right) \sin\left(\frac{3\pi}{n}\right)} = \frac{1}{\sin\left(\frac{2\pi}{n}\right)}$$

$$\Rightarrow 2 \cos\left(\frac{2\pi}{n}\right) \sin\left(\frac{2\pi}{n}\right) = \sin\left(\frac{3\pi}{n}\right)$$

$$\Rightarrow \sin\left(\frac{4\pi}{n}\right) = \sin\left(\frac{3\pi}{n}\right)$$

$$\Rightarrow \sin\left(\frac{4\pi}{n}\right) = \sin\left(\pi - \frac{3\pi}{n}\right)$$

$$\Rightarrow \left(\frac{4\pi}{n}\right) = \left(\pi - \frac{3\pi}{n}\right)$$

$$\Rightarrow \left(\frac{7\pi}{n}\right) = \pi$$

$$\Rightarrow n = 7$$

134. Let r be the radius of the in-circle and r_1, r_2 and r_3 are the ex-radii of the given triangle.

$$\begin{aligned} \text{Then } \frac{1}{\sqrt{A_1}} + \frac{1}{\sqrt{A_2}} + \frac{1}{\sqrt{A_3}} &= \frac{1}{\sqrt{\pi r_1^2}} + \frac{1}{\sqrt{\pi r_2^2}} + \frac{1}{\sqrt{\pi r_3^2}} \\ &= \frac{1}{\sqrt{\pi}} \left(\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}\right) \\ &= \frac{1}{\sqrt{\pi}} \times \frac{1}{r} \\ &= \frac{1}{\sqrt{\pi r^2}} \\ &= \frac{1}{\sqrt{A}} \end{aligned}$$

Hence, the result.

135. Let the perimeter of the polygon of n sides = nx

Let A_1 = the area of the polygon of n sides

$$= nx^2 \cot\left(\frac{\pi}{n}\right)$$

and A_2 = the area of the circle = πx^2

$$\text{Now, } \frac{A_2}{A_1} = \frac{\pi x^2}{nx^2 \cot\left(\frac{\pi}{n}\right)} = \frac{\tan\left(\frac{\pi}{n}\right)}{\frac{\pi}{n}}$$

$$\text{Thus, } A_2 : A_1 = \tan\left(\frac{\pi}{n}\right) : \left(\frac{\pi}{n}\right)$$

136. We have

$$\begin{aligned} r + R &= \frac{a}{2} \cot\left(\frac{a}{n}\right) + \frac{a}{2} \operatorname{cosec}\left(\frac{a}{n}\right) \\ &= \frac{a}{2} \left[\cot\left(\frac{a}{n}\right) + \operatorname{cosec}\left(\frac{a}{n}\right) \right] \\ &= \frac{a}{2} \left[\frac{\cos\left(\frac{a}{n}\right)}{\sin\left(\frac{a}{n}\right)} + \frac{1}{\sin\left(\frac{a}{n}\right)} \right] \\ &= \frac{a}{2} \left[\frac{1 + \cos\left(\frac{a}{n}\right)}{\sin\left(\frac{a}{n}\right)} \right] \\ &= \frac{a}{2} \left[\frac{2 \cos^2\left(\frac{a}{2n}\right)}{2 \sin\left(\frac{a}{2n}\right) \cos\left(\frac{a}{2n}\right)} \right] \\ &= \frac{a}{2} \cot\left(\frac{a}{2n}\right) \end{aligned}$$