EXERCISES

LEVEL I

(Problems Based on Fundamentals)

SINE RULE

1. If in a triangle ABC, $\frac{a^2-b^2}{a^2+b^2}=\frac{\sin(A-B)}{\sin(A+B)}$, then the triangle is right angled or isosceles?
2. Prove that $\frac{\sin(B-C)}{\sin(B+C)}=\frac{b^2-c^2}{a^2}$
3. In a ΔABC such that $\angle A = 45°$ and $\angle B = 75°$ then find $a+c\sqrt{2}$.
4. Prove that
$$\frac{a^2\sin(B-C)}{\sin B+\sin C}+\frac{b^2\sin(C-A)}{\sin C+\sin A}+\frac{c^2\sin(A-B)}{\sin A+\sin B}=0$$
5. Prove that
$$\frac{b^2-c^2}{\cos B+\cos C}+\frac{c^2-a^2}{\cos C+\cos A}+\frac{a^2-b^2}{\cos A+\cos B}=0$$
6. In a triangle ABC, if a^2, b^2, c^2 are in AP then prove that $\cot A$, $\cot B$, $\cot C$ are in AP.
7. If $\cot\frac{A}{2}=\frac{b+c}{a}$, then prove that ΔABC is right angled.
8. Prove that a^2, b^2, c^2 are in A.P., if $\frac{\sin A}{\sin C}=\frac{\sin(A-B)}{\sin(B-C)}$
9. In any ΔABC, prove that
$$\Pi\left(\frac{\sin^2 A+\sin A+1}{\sin A}\right)>27.$$
10. In a triangle ABC, prove that,
$$a\sin\left(\frac{A}{2}+B\right)=(b+c)\sin\left(\frac{A}{2}\right)$$
11. In a triangle ABC, prove that,
$$\frac{a\sin(B-C)}{b^2-c^2}=\frac{b\sin(C-A)}{c^2-a^2}=\frac{c\sin(A-B)}{a^2-b^2}$$
12. Prove that $\frac{1+\cos(A-B)\cos C}{1+\cos(A-C)\cos B}=\frac{a^2+b^2}{a^2+c^2}$
13. In a ΔABC, if $\cos A + 2\cos B + \cos C = 2$ then prove that the sides of a triangle are in AP
14. In a ΔABC, if
$\cos A\cos B+\sin A\sin B\sin C=1$,
then prove that $a:b:c=1:1:\sqrt{2}$.

COSINE RULE

15. In a ΔABC, prove that
$$a(b\cos C-c\cos B)=b^2-c^2$$
16. Prove that
$$\frac{\cos A}{a}+\frac{\cos B}{b}+\frac{\cos C}{c}=\frac{a^2+b^2+c^2}{2abc}$$
17. Prove that
$$(a-b)^2\cos^2\left(\frac{C}{2}\right)+(a+b)^2\sin^2\left(\frac{C}{2}\right)=c^2$$
18. In a ΔABC, if
$(a+b+c)(a-b+c)=3ac$, then find $\angle B$
19. In any ΔABC, if $2\cos B=\frac{a}{c}$
prove that the triangle is isosceles.
20. In a ΔABC, if $(a+b+c)(b+c-a)=\lambda bc$ then find the value of λ.
21. If the angles A, B, C of a triangle are in AP and its sides a, b, c are in GP, prove that a^2, b^2, c^2 are in AP.
22. If the line segment joining the points $P(a_1, b_1)$ and $Q(a_2, b_2)$ subtends an angle θ at the origin, prove that
$$\cos\theta=\left(\frac{a_1a_2+b_1b_2}{\sqrt{a_1^2+b_1^2}\sqrt{a_2^2+b_2^2}}\right)$$
23. In a triangle ABC, if $\cot A$, $\cot B$, $\cot C$ are in AP, prove that a^2, b^2, c^2 are in AP.
24. If the sides of a triangle are a, b and $\sqrt{a^2+ab+b^2}$ then find its greatest angle.
25. In a triangle ABC, if $a\cos A = b\cos B$, then prove that triangle is right angled isosceles.
26. In a triangle ABC, the angles are in AP, then prove that,
$$2\cos\left(\frac{A-C}{2}\right)=\frac{a+c}{\sqrt{a^2-ac+c^2}}$$
27. In a triangle ABC, prove that
$$\left(\frac{b^2-c^2}{a^2}\right)\sin 2A+\left(\frac{c^2-a^2}{b^2}\right)\sin 2B+\left(\frac{a^2-b^2}{c^2}\right)\sin 2C=0$$
28. In a triangle ABC, if $\angle A = 60°$, then find the value of $\left(1+\frac{a}{c}+\frac{b}{c}\right)\left(1+\frac{c}{b}-\frac{a}{b}\right)$.
29. If $\frac{1}{a+c}+\frac{1}{b+c}=\frac{3}{a+b+c}$, then find $\angle C$
30. If in a triangle ABC,
$$\frac{2\cos A}{a}+\frac{2\cos B}{b}+\frac{2\cos C}{c}=\frac{1}{bc}+\frac{b}{ca},$$
then find the angle A in degrees.

PROJECTION RULE

31. In any ΔABC, prove that
$$2\left(a\sin^2\frac{C}{2}+c\sin^2\frac{A}{2}\right)=a+c-b$$

32. In any ΔABC, prove that
$$2\left(b\cos^2\frac{C}{2}+c\cos^2\frac{B}{2}\right)=a+b+c$$

33. In any ΔABC, prove that
$(b+c)\cos A+(c+a)\cos B+(a+b)\cos C$
$=(a+b+c)$

34. In a ΔABC, prove that, $\dfrac{\sin B}{\sin C}=\dfrac{c-a\cos B}{b-a\cos C}$

35. In any ΔABC, prove that
$$2\left(a\sin^2\frac{C}{2}+c\sin^2\frac{A}{2}\right)=a+c-b$$

36. In any ΔABC, prove that
$$\frac{\cos A}{b\cos C+c\cos B}+\frac{\cos B}{c\cos A+a\cos C}+\frac{\cos C}{a\cos B+b\cos A}=\frac{a^2+b^2+c^2}{2abc}$$

37. In any ΔABC prove that $2(bc\cos A+ca\cos B+ab\cos C)=(a^2+b^2+c^2)$

NAPIER ANALOGY

38. In any ΔABC, $b=\sqrt{3}+1, c=\sqrt{3}-1$ and $\angle A=60°$ then find the value of $\tan\left(\dfrac{B-C}{2}\right)$.

39. In any ΔABC, $b=\sqrt{3}, c=1, B-C=90°$ then find $\angle A$.

40. If in a ΔABC, $a=6$, $b=3$ and $\cos(A-B)=\dfrac{4}{5}$, then find the angle C.

41. In a ΔABC, if
$$x=\tan\left(\frac{B-C}{2}\right)\tan\left(\frac{A}{2}\right),$$
$$y=\tan\left(\frac{C-A}{2}\right)\tan\left(\frac{B}{2}\right)\text{ and}$$
$$z=\tan\left(\frac{A-B}{2}\right)\tan\left(\frac{C}{2}\right),$$
then prove that $x+y+z+xyz=0$

42. In a ΔABC, if $a=5$, $b=4$ and $\cos(A-B)=\dfrac{31}{32}$, then prove that $c=6$.

HALF ANGLED FORMULA

43. In a ΔABC, if $a=13$, $b=14$ and $c=15$, then find the value of
(i) $\sin\dfrac{A}{2}$
(ii) $\cos\dfrac{B}{2}$
(iii) $\cos A$

44. In a ΔABC, if $\cos\left(\dfrac{A}{2}\right)=\sqrt{\dfrac{b+c}{2c}}$, prove that ΔABC is right angled at C.

45. In a ΔABC, prove that,
$$b\cos^2\left(\frac{C}{2}\right)+c\cos^2\left(\frac{B}{2}\right)=s$$

46. In a ΔABC, prove that
$$bc\cos^2\left(\frac{A}{2}\right)+ca\cos^2\left(\frac{B}{2}\right)+ab\cos^2\left(\frac{C}{2}\right)=s^2$$

47. In a ΔABC, prove that
$$2ac\sin\left(\frac{A-B+C}{2}\right)=(a^2+c^2-b^2).$$

48. In a ΔABC, $3a=b+c$, then find the value of $\cot\left(\dfrac{B}{2}\right)\cot\left(\dfrac{C}{2}\right)$

49. In a ΔABC, prove that
$$1-\tan\left(\frac{A}{2}\right)\tan\left(\frac{B}{2}\right)=\frac{2c}{(a+b+c)}$$

50. In a ΔABC, prove that:
$$\cot\left(\frac{A}{2}\right)+\cot\left(\frac{B}{2}\right)+\cot\left(\frac{C}{2}\right)=\left(\frac{a+b+c}{b+c-a}\right)\cot\left(\frac{A}{2}\right)$$

51. In a ΔABC, if $\cot\left(\dfrac{A}{2}\right), \cot\left(\dfrac{B}{2}\right), \cot\left(\dfrac{C}{2}\right)$ are in AP, then prove that a, b, c are in AP

52 In a ΔABC, $c(a+b)\cos\dfrac{B}{2}$
$=b(a+c)\cos\dfrac{C}{2}$,
then prove that the triangle is isosceles.

53. In a ΔABC, prove that
$$\frac{\cot\left(\frac{A}{2}\right)+\cot\left(\frac{B}{2}\right)+\cot\left(\frac{C}{2}\right)}{\cot A+\cot B+\cot C}=\frac{(a+b+c)^2}{a^2+b^2+c^2}$$

54. In a ΔABC, if
$$c(a+b)\cos\left(\frac{B}{2}\right)=b(a+c)\cos\left(\frac{C}{2}\right)$$
then prove that the triangle ABC is isosceles.

AREA OF A TRIANGLE

55. In any ΔABC, if $a=\sqrt{2}, b=\sqrt{3}$ and $c=\sqrt{5}$, then find the area of the ΔABC.

(1)

$\Rightarrow \quad \frac{a^2-b^2}{a^2+b^2}=\frac{\sin^2 A-\sin^2 B}{\sin^2 C}$

$\Rightarrow \quad \frac{a^2-b^2}{a^2+b^2}=\frac{k^2(a^2-b^2)}{k^2c^2}$

$\Rightarrow \quad \frac{a^2-b^2}{a^2+b^2}=\frac{(a^2-b^2)}{c^2}$

$\Rightarrow \quad (a^2-b^2)\left(\frac{1}{a^2+b^2}-\frac{1}{c^2}\right)=0$

$\Rightarrow \quad (a^2-b^2)=0, \left(\frac{1}{a^2+b^2}-\frac{1}{c^2}\right)=0$

$\Rightarrow \quad a^2=b^2, \frac{1}{a^2+b^2}=\frac{1}{c^2}$

$\Rightarrow \quad a=b, a^2+b^2=c^2$

Thus, t hetriangle is isosceles or right angled.

2. We have

$$\frac{\sin(B-C)}{\sin(B+C)}=\frac{\sin(B-C)}{\sin(B+C)}\times\frac{\sin(B+C)}{\sin(B+C)}$$
$$=\frac{\sin^2(B)-\sin^2(C)}{\sin^2(B+C)}$$
$$=\frac{\sin^2(B)-\sin^2(C)}{\sin^2(\pi-A)}$$
$$=\frac{\sin^2(B)-\sin^2(C)}{\sin^2(A)}$$
$$=\frac{k^2b^2-k^2c^2}{k^2a^2}$$
$$=\frac{b^2-c^2}{a^2}$$

3. We have

$\angle C = 180° - (A+B)$
$= 180° - (45° + 75°)$
$= 180° - 120°$
$= 60°$

From t hesine rule, we can write

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$$

$\Rightarrow \quad \frac{a}{\sin(45°)}=\frac{b}{\sin(75°)}=\frac{c}{\sin(60°)}$

$\Rightarrow \quad \frac{a}{\frac{1}{\sqrt{2}}}=\frac{b}{\frac{\sqrt{3}+1}{2\sqrt{2}}}=\frac{c}{\frac{\sqrt{3}}{2}}=k\text{(say)}$

Now, $a+c\sqrt{2}$

$$=\frac{k}{\sqrt{2}}+\left(\frac{k\sqrt{3}}{2}\right)\sqrt{2}$$
$$=\frac{k}{\sqrt{2}}+\left(\frac{k\sqrt{3}}{\sqrt{2}}\right)$$
$$=\left(\frac{\sqrt{3}+1}{\sqrt{2}}\right)k$$
$$=2\left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)k$$
$$=2b$$

4. Now,

$$\frac{a^2\sin(B-C)}{\sin B+\sin C}=\frac{ak\sin A\sin(B-C)}{\sin B+\sin C}$$
$$=\frac{ak\sin(B+C)\sin(B-C)}{\sin B+\sin C}$$
$$=ak\left(\frac{\sin^2(B)-\sin^2(C)}{\sin B+\sin C}\right)$$
$$=ak(\sin B-\sin C)$$
$$=k\sin A(\sin B-\sin C)$$
$$=k(\sin A\sin B-\sin A\sin C)$$

Similarly,

$$\frac{b^2\sin(C-A)}{\sin C+\sin A}=k(\sin B\sin C-\sin A\sin B)$$

and $\frac{c^2\sin(A-B)}{\sin A+\sin B}=k(\sin A\sin C-\sin C\sin B)$

Thus, LHS

$= k[\sin A\sin B-\sin A\sin C+\sin B\sin C$
$-\sin A\sin B+\sin A\sin C-\sin C\sin B]$
$= 0$

5. Now,

$$\frac{b^2-c^2}{\cos B+\cos C}=\frac{k^2(\sin^2 B-\sin^2 C)}{\cos B+\cos C}$$
$$=\frac{k^2(1-\cos^2 B-1+\cos^2 C)}{\cos B+\cos C}$$
$$=-\frac{k^2(\cos^2 B-\cos^2 C)}{\cos B+\cos C}$$
$$=-k^2(\cos B-\cos C)$$

Similarly,

$$\frac{c^2-a^2}{\cos C+\cos A}=-k^2(\cos C-\cos A)$$

and $\frac{a^2-b^2}{\cos A+\cos B}=-k^2(\cos A-\cos B)$

Thus, LHS

$=-k^2[\cos B-\cos C+\cos C-\cos A]$

$\cos A-\cos B$

$= 0$

6. Given,

a^2, b^2, c^2 are in AP

$\Rightarrow$ $b^2 - a^2 = c^2 - b^2$

$\Rightarrow$ $\sin^2 B - \sin^2 A = \sin^2 C - \sin^2 B$

$\Rightarrow$ $\sin(B + A)\sin(B - A)$

$= \sin(C + B)\sin(C - B)$

$\Rightarrow$ $\sin C(\sin B\cos A - \cos B\sin A)$

$= \sin A(\sin C\cos B - \cos C\sin B)$

Dividing both the sides by $\sin A\sin B\sin C$, we get,

$\Rightarrow$ $\cot A - \cot B = \cot B - \cot C$

$\Rightarrow$ $\cot A, \cot B, \cot C$ are in AP

7. We have

$$\cot\left(\frac{A}{2}\right) = \frac{b+c}{a}$$

$$\Rightarrow \cot\left(\frac{A}{2}\right) = \frac{\sin B + \sin C}{\sin A}$$

$$\Rightarrow \cot\left(\frac{A}{2}\right) = \frac{2\sin\left(\frac{B+C}{2}\right)\cos\left(\frac{B-C}{2}\right)}{\sin A}$$

$$\Rightarrow \cot\left(\frac{A}{2}\right) = \frac{2\sin\left(\frac{\pi}{2} - \frac{A}{2}\right)\cos\left(\frac{B-C}{2}\right)}{2\sin\left(\frac{A}{2}\right)\cos\left(\frac{A}{2}\right)}$$

$$\Rightarrow \cot\left(\frac{A}{2}\right) = \frac{2\cos\left(\frac{A}{2}\right)\cos\left(\frac{B-C}{2}\right)}{2\sin\left(\frac{A}{2}\right)\cos\left(\frac{A}{2}\right)}$$

$$\Rightarrow \cot\left(\frac{A}{2}\right) = \frac{\cos\left(\frac{B-C}{2}\right)}{\sin\left(\frac{A}{2}\right)}$$

$$\Rightarrow \frac{\cos\left(\frac{A}{2}\right)}{\sin\left(\frac{A}{2}\right)} = \frac{\cos\left(\frac{B-C}{2}\right)}{\sin\left(\frac{A}{2}\right)}$$

$$\Rightarrow \cos\left(\frac{A}{2}\right) = \cos\left(\frac{B-C}{2}\right)$$

$$\Rightarrow \frac{A}{2} = \frac{B-C}{2}$$

$\Rightarrow$ $A + C = B$

$\Rightarrow$ $2B = A + B + C = 180°$

$\Rightarrow$ $B = 90°$

Thus, the triangle is right angled.

8. Given,

a^2, b^2, c^2 are in AP.

$b^2 - a^2 = c^2 - b^2$

$\sin^2 B - \sin^2 C = \sin^2 C - \sin^2 B$

$\sin(B + A)\sin(B - A) = \sin(C + B)\sin(C - B)$

$\sin(C)\sin(B - A) = \sin(A)\sin(C - B)$

$$\frac{\sin(B-A)}{\sin(C-B)} = \frac{\sin(A)}{\sin(C)}$$

$$\frac{\sin(A)}{\sin(C)} = \frac{\sin(A-B)}{\sin(B-C)}$$

9. We have

$$\prod\left(\frac{\sin^2 A + \sin A + 1}{\sin A}\right) = \left(\sin A + \frac{1}{\sin A} + 1\right)\left(\sin B + \frac{1}{\sin B} + 1\right)\left(\sin C + \frac{1}{\sin C} + 1\right)$$

$> (2 + 1)(2 + 1)(2 + 1) = 27$

(applying AM ≥ GM)

10. Let $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = k$

We have,

$$\frac{b+c}{a} = \frac{k(\sin B + \sin C)}{k\sin A}$$

$$= \frac{(\sin B + \sin C)}{\sin A}$$

$$= \frac{2\sin\left(\frac{B+C}{2}\right)\cos\left(\frac{B-C}{2}\right)}{2\sin\left(\frac{A}{2}\right)\cos\left(\frac{A}{2}\right)}$$

$$= \frac{2\sin\left(\frac{\pi}{2} - \frac{A}{2}\right)\cos\left(\frac{B-C}{2}\right)}{2\sin\left(\frac{A}{2}\right)\cos\left(\frac{A}{2}\right)}$$

$$= \frac{2\cos\left(\frac{A}{2}\right)\cos\left(\frac{B-C}{2}\right)}{2\sin\left(\frac{A}{2}\right)\cos\left(\frac{A}{2}\right)}$$

$$= \frac{\cos\left(\frac{B-C}{2}\right)}{\sin\left(\frac{A}{2}\right)}$$

$$= \frac{\sin\left(\frac{A}{2} + B\right)}{\sin\left(\frac{A}{2}\right)}$$

11. Let $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = k$

We have, $\dfrac{a\sin(B-C)}{b^2 - c^2}$

$$= \frac{k \sin A \cdot \sin (C - A)}{k^2(\sin^2 C - \sin^2 A)}$$

$$= \frac{\sin (B + C) \cdot \sin (B - C)}{k(\sin^2 B - \sin^2 C)}$$

$$= \frac{(\sin^2 B - \sin^2 C)}{k(\sin^2 B - \sin^2 C)} = \frac{1}{k} \frac{b \sin (C - A)}{c^2 - a^2}$$

$$= \frac{k \sin (C + A) \cdot \sin (C - A)}{k^2(\sin^2 C - \sin^2 A)}$$

$$= \frac{\sin^2 C - \sin^2 A}{k(\sin^2 C - \sin^2 A)} = \frac{1}{k}$$

Also,

$$\frac{c \sin (A - B)}{a^2 - b^2} = \frac{k \sin C \sin (A - B)}{k^2(\sin^2 A - \sin^2 B)}$$

$$= \frac{\sin (A + B) \sin (A - B)}{k(\sin^2 A - \sin^2 B)}$$

$$= \frac{\sin^2 A - \sin^2 B}{k(\sin^2 A - \sin^2 B)} = \frac{1}{k}$$

Hence, the result.

12. Given,

$$\frac{1 + \cos (A - B) \cos C}{1 + \cos (A - C) \cos B}$$

$$= \frac{1 + \cos (A - B) \cos (\pi - (A + B))}{1 + \cos (A - C) \cos (\pi - (A + C))}$$

$$= \frac{1 - \cos (A - B) \cos (A + B)}{1 - \cos (A - C) \cos (A + C)}$$

$$= \frac{1 - \{\cos^2 A - \sin^2 B\}}{1 - \{\cos^2 A - \sin^2 C\}}$$

$$= \frac{\sin^2 A + \sin^2 B}{\sin^2 A + \sin^2 C}$$

$$= \frac{a^2 + b^2}{a^2 + c^2}$$

13. Given,

$$\cos A + 2\cos B + \cos C = 2$$

$$\Rightarrow \cos A + \cos C = 2(1 - \cos B)$$

$$\Rightarrow 2 \cos\left(\frac{A+C}{2}\right) \cos\left(\frac{A-C}{2}\right) = 4 \sin^2\left(\frac{B}{2}\right)$$

$$\Rightarrow \cos\left(\frac{\pi}{2} - \frac{B}{2}\right) \cos\left(\frac{A-C}{2}\right) = 2 \sin^2\left(\frac{B}{2}\right)$$

$$\Rightarrow \sin\left(\frac{B}{2}\right) \cos\left(\frac{A-C}{2}\right) = 2 \sin^2\left(\frac{B}{2}\right)$$

$$\Rightarrow \cos\left(\frac{A-C}{2}\right) = 2 \sin\left(\frac{B}{2}\right)$$

Multiplying both sides by $2 \cos\left(\frac{B}{2}\right)$, we get,

$$\Rightarrow 2 \cos\left(\frac{B}{2}\right) \cos\left(\frac{A-C}{2}\right) = 2\left(2 \cos\left(\frac{B}{2}\right) \sin\left(\frac{B}{2}\right)\right)$$

$$\Rightarrow 2 \sin\left(\frac{A+C}{2}\right) \cos\left(\frac{A-C}{2}\right) = 2 \sin B$$

$$\Rightarrow \sin A + \sin C = 2 \sin B$$

$$\Rightarrow a + c = 2b$$

$\Rightarrow$ a, b, c are in AP

14 We have

$$\cos A + \cos B + \sin A \sin B \sin C = 1$$

$$\Rightarrow \frac{1 - \cos A \cos B}{\sin A \sin B} = \sin C$$

$$\Rightarrow \frac{1 - \cos A \cos B}{\sin A \sin B} = \sin C \le 1$$

$$\Rightarrow 1 - \cos A \cos B \le \sin A \sin B$$

$$\Rightarrow 1 \le \cos A \cos B - \sin A \sin B$$

$$\Rightarrow \cos (A - B) \ge 1$$

$$\Rightarrow \cos (A - B) = 1$$

$$\Rightarrow \cos (A - B) = \cos (0)$$

$$\Rightarrow A - B = 0$$

$$\Rightarrow A = B$$

Therefore,

$$\sin C = \frac{1 - \cos A \cos B}{\sin A \sin B}$$

$$= \frac{1 - \cos A \cos A}{\sin A \sin A} = \frac{1 - \cos^2 A}{\sin^2 A} = \frac{\sin^2 A}{\sin^2 A} = 1$$

$$\Rightarrow C = 90°$$

Hence, $A = 45° = B$, $C = 90°$

Now, $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

$$\Rightarrow \frac{a}{\sin 45°} = \frac{b}{\sin 45°} = \frac{c}{\sin 90°}$$

$$\Rightarrow \frac{a}{\frac{1}{\sqrt{2}}} = \frac{b}{\frac{1}{\sqrt{2}}} = \frac{c}{1}$$

$$\Rightarrow \frac{a}{1} = \frac{b}{1} = \frac{c}{\sqrt{2}}.$$

$$\Rightarrow a:b:c = 1:1:\sqrt{2}$$

Hence, the result.

15. We have,

$a(b \cos C - c \cos B)$

$$= (ab \cos C - ac \cos B)$$

$$= ab\left(\frac{a^2 + b^2 - c^2}{2ab}\right) - ac\left(\frac{a^2 + c^2 - b^2}{2ac}\right)$$

$$= \left(\frac{a^2 + b^2 - c^2}{2}\right) - \left(\frac{a^2 + c^2 - b^2}{2}\right)$$

$$= \frac{1}{2}(a^2 + b^2 - c^2 - a^2 - c^2 + b^2)$$

$$= \frac{1}{2}(b^2 - c^2 - c^2 + b^2)$$

$$= \frac{1}{2}(2b^2 - 2c^2)$$

$$= (b^2 - c^2)$$

Hence, the result.

16. We have $\frac{\cos A}{a}+\frac{\cos B}{b}+\frac{\cos C}{c}$

$$=\frac{1}{2abc}(b^2+c^2-a^2+a^2+c^2-b^2+a^2+b^2-c^2)$$

$$=\frac{(a^2+b^2+c^2)}{2abc}$$

17. We have

$$(a-b)^2\cos^2\left(\frac{C}{2}\right)+(a+b)^2\sin^2\left(\frac{C}{2}\right)$$

$$=(a^2+b^2)\left(\cos^2\left(\frac{C}{2}\right)+\sin^2\left(\frac{C}{2}\right)\right)$$

$$-2ab\left(\cos^2\left(\frac{C}{2}\right)-\sin^2\left(\frac{C}{2}\right)\right)$$

$$=(a+b^2)-2ab\cos C$$

$$=(a+b^2)-2ab\left(\frac{a^2+b^2-c^2}{2ab}\right)$$

$$=(a+b^2)-(a^2+b^2-c^2)$$

$$=c^2$$

18. We have

$(a+b+c)(a-b+c)=3ac$

$\Rightarrow$ $(a+c)^2-b^2=3ac$

$\Rightarrow$ $a^2+c^2-b^2=3ac-2ac=ac$

$\Rightarrow$ $\frac{a^2+c^2-b^2}{2ac}=\frac{ac}{2ac}=\frac{1}{2}$

$\Rightarrow$ $\cos B=\frac{1}{2}$

$\Rightarrow$ $B=\frac{\pi}{3}$

Hence, the angle B is 60°.

19. We have $2\cos B=\frac{a}{c}$

$\Rightarrow$ $2\left(\frac{a^2+c^2-b^2}{2ac}\right)=\frac{a}{c}$

$\Rightarrow$ $\left(\frac{a^2+c^2-b^2}{a}\right)=a$

$\Rightarrow$ $(a^2+c^2-b^2)=a^2$

$\Rightarrow$ $(c^2-b^2)=0$

$\Rightarrow$ $c^2=b^2$

$\Rightarrow$ $c=b$

Thus, the triangle is isosceles.

20. Given, $(a+b+c)(b+c-a)=\lambda bc$

$\{(b+c)^2-a^2\}=\lambda bc$

$(b^2+c^2-a^2)=(\lambda-2)bc$

$$\frac{(b^2+c^2-a^2)}{2bc}=\frac{(\lambda-2)bc}{2bc}=\frac{\lambda-2}{2}$$

$$\cos A=\frac{\lambda-2}{2}$$

$$-1\le\frac{\lambda-2}{2}\le 1$$

$$-2\le\lambda-2\le 2$$

$$0\le\lambda\le 4$$

21. Given, A, B, C are in AP

$2B=A+C$

$3B=A+B+C=\pi$

$B=\frac{\pi}{3}$

$\cos B=\cos\left(\frac{\pi}{3}\right)=\frac{1}{2}$

$\frac{a^2+c^2-b^2}{2ac}=\frac{1}{2}$

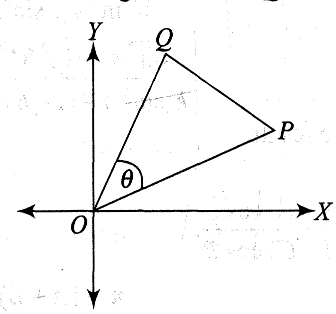
$a^2+c^2-b^2=ac$

$a^2+c^2-b^2=b^2$

$2b^2=a^2+c^2$

$a^2, b^2, c^2\in$ AP

22. Given, O be the origin and $\angle POQ=\theta$



Now, $\cos\theta=\frac{OP^2+OQ^2-PQ^2}{2OP\cdot OQ}$

$$\cos\theta=\frac{(a_1^2+b_1^2)+(a_2^2+b_2^2)-\{(a_1-a_2)^2+(b_1-b_2)^2\}}{2\sqrt{a_1^2+b_1^2}\cdot\sqrt{a_2^2+b_2^2}}$$

$$=\frac{2(a_1a_2+b_1b_2)}{2\sqrt{a_1^2+b_1^2}\cdot\sqrt{a_2^2+b_2^2}}$$

$$=\frac{(a_1a_2+b_1b_2)}{\sqrt{a_1^2+b_1^2}\cdot\sqrt{a_2^2+b_2^2}}$$

23. Given, $\cot A, \cot B, \cot C$ are in AP

$\frac{\cos A}{\sin A},\frac{\cos B}{\sin B},\frac{\cos C}{\sin C}\in$ AP

$\frac{b^2+c^2-a^2}{2abck},\frac{a^2+c^2-b^2}{2abck},\frac{a^2+b^2-c^2}{2abck}\in$ AP

$(b^2+c^2-a^2), (a^2+c^2-b^2), (a^2+b^2-c^2)\in$ AP

Subtracting $(a^2+b^2+c^2)$ to each term

$(-2a^2), (-2b^2), (-2c^2)\in$ AP

$a^2, b^2, c^2\in$ AP

24. Let $c=\sqrt{a^2+ab+b^2}$

Clearly, side c is the greatest

Thus, the angle C is the greatest.

Now, $\cos C = \dfrac{a^2+b^2-c^2}{2ab}$

$= \dfrac{a^2+b^2-(a^2+ab+b^2)}{2ab}$

$= -\dfrac{ab}{2ab} = -\dfrac{1}{2}$

Thus, $\angle C = \dfrac{2\pi}{3}$

25. We have $a\cos A = b\cos B$

$a\left(\dfrac{b^2+c^2-a^2}{2bc}\right) = b\left(\dfrac{a^2+c^2-b^2}{2ac}\right)$

$\Rightarrow\quad a\left(\dfrac{b^2+c^2-a^2}{b}\right) = b\left(\dfrac{a^2+c^2-b^2}{a}\right)$

$\Rightarrow\quad a^2(b^2+c^2-a^2) = b^2(a^2-c^2-b^2)$

$\Rightarrow\quad a^2(c^2-a^2) = b^2(c^2-b^2)$

$\Rightarrow\quad c^2(a^2-b^2) = (a^4-b^4)$

$\Rightarrow\quad c^2(a^2-b^2) = (a^2-b^2)(a^2+b^2)$

$\Rightarrow\quad (a^2-b^2)((a^2+b^2-c^2) = 0$

$\Rightarrow\quad (a^2-b^2) = 0, (a^2+b^2) = c^2$

$\Rightarrow\quad a = b, (a^2+b^2) = c^2$

Thus, the triangle is right angled isosceles.

26. Since the angles are in AP, so $A + C = 2B$

$\Rightarrow\quad A + B + C = 3B$

$\Rightarrow\quad 3B = 180°$

$\Rightarrow\quad B = 60°$

$\Rightarrow\quad \cos B = \cos(60°) = \dfrac{1}{2}$

$\Rightarrow\quad \dfrac{a^2+c^2-b^2}{2ac} = \dfrac{1}{2}$

$\Rightarrow\quad a^2+c^2-b^2 = ac$

$\Rightarrow\quad a^2+c^2-ac = b^2$

Now, RHS

$= \dfrac{a+c}{\sqrt{a^2-ac+c^2}}$

$= \dfrac{a+c}{b}$

$= \dfrac{k(\sin A+\sin C)}{k\sin B}$

$= \dfrac{2\sin\left(\frac{A+C}{2}\right)\cos\left(\frac{A-C}{2}\right)}{2\sin\left(\frac{B}{2}\right)\cos\left(\frac{B}{2}\right)}$

$= \dfrac{\cos\left(\frac{B}{2}\right)\cos\left(\frac{A-C}{2}\right)}{\sin\left(\frac{B}{2}\right)\cos\left(\frac{B}{2}\right)}$

$= \dfrac{\cos\left(\frac{A-C}{2}\right)}{\sin\left(\frac{B}{2}\right)}$

$= \dfrac{\cos\left(\frac{A-C}{2}\right)}{\sin\left(\frac{60°}{2}\right)}$

$= 2\cos\left(\dfrac{A-C}{2}\right)$

27. We have

$\left(\dfrac{b^2-c^2}{a^2}\right)\sin 2A = \left(\dfrac{b^2-c^2}{a^2}\right)(2\sin A\cdot\cos A)$

$= \left(\dfrac{b^2-c^2}{a^2}\right)\left(2ka\left(\dfrac{b^2+c^2-a^2}{2bc}\right)\right)$

$= \left(\dfrac{(b^2-c^2)(b^2+c^2-a^2)}{kabc}\right)$

$= \dfrac{1}{kabc}\times\{(b^4-c^4)-a^2(b^2-c^2)\}$

Similarly,

$\left(\dfrac{c^2-a^2}{b^2}\right)\times\sin 2B = \dfrac{1}{kabc}\times\{(c^4-a^4)-b^2(c^2-a^2)\}$

$\left(\dfrac{a^2-b^2}{c^2}\right)\sin 2C$ and

$= \dfrac{1}{kabc}\times\{(a^4-b^4)-c^2(a^2-b^2)\}$

Thus, $\left(\dfrac{b^2-c^2}{a^2}\right)\sin 2A + \left(\dfrac{c^2-a^2}{b^2}\right)\sin 2B + \left(\dfrac{a^2-b^2}{c^2}\right)\sin 2C$

$= \dfrac{1}{kabc}\times\{(b^4-c^4)-a^2(b^2-c^2)\}$

$+ \dfrac{1}{kabc}\times\{(c^4-a^4)-b^2(c^2-a^2)\}$

$+ \dfrac{1}{kabc}\times\{(a^4-b^4)-c^2(a^2-b^2)\}$

$= \dfrac{1}{kabc}\times[(b^4-c^4+c^4-a^4+a^4-b^4)$

$= a^2(c^2-b^2)+b^2(a^2-c^2)+c^2(b^2-a^2)]$

$= 0$

28. Given,

$\angle A = 60°$

$\Rightarrow\quad \cos A = \cos(60°)$

$\Rightarrow\quad \cos A = \dfrac{1}{2}$

$\Rightarrow \quad \frac{b^2+c^2-a^2}{2bc}=\frac{1}{2}$

$\Rightarrow \quad (b^2+c^2-a^2)=b$ ...(i)

Now,

$$\left(1+\frac{a}{c}+\frac{b}{c}\right)\left(1+\frac{c}{b}-\frac{a}{b}\right)=\left(\frac{c+a+b}{c}\right)\left(\frac{b+c-a}{b}\right)$$

$$=\left(\frac{b+c+a}{c}\right)\left(\frac{b+c-a}{b}\right)$$

$$=\left(\frac{(b+c)^2-a^2}{bc}\right)$$

$$=\left(\frac{b^2+c^2-a^2+2bc}{bc}\right)$$

$$=\left(\frac{bc+2bc}{bc}\right), \text{ from (i)}$$

$$=\left(\frac{3bc}{bc}\right)$$

$$=3$$

29. Given, $\frac{1}{a+c}+\frac{1}{b+c}=\frac{3}{a+b+c}$

$\Rightarrow \quad \frac{a+b+c}{a+c}+\frac{a+b+c}{b+c}=3$

$\Rightarrow \quad 1+\frac{b}{a+c}+1+\frac{a}{b+c}=3$

$\Rightarrow \quad \frac{b}{a+c}+\frac{a}{b+c}=1$

$\Rightarrow \quad b(b+c)+a(a+c)=(a+c)(b+c)$

$\Rightarrow \quad b^2+bc+a^2+ac=ab+ac+bc+c^2$

$\Rightarrow \quad a^2+b^2-c^2=ab$

Now, $\cos(C)=\left(\frac{a^2+b^2-c^2}{2ab}\right)$

$$=\frac{ab}{2ab}=\frac{1}{2}$$

$\Rightarrow \quad C=\frac{\pi}{3}$

30. We have

$$\frac{2\cos A}{a}+\frac{2\cos B}{b}+\frac{2\cos C}{c}=\frac{1}{bc}+\frac{b}{ca}$$

$\Rightarrow \quad \frac{2\,bc\cos A}{abc}+\frac{ac\cos B}{abc}+\frac{2bc\cos C}{abc}$

$$=\frac{a^2}{abc}+\frac{b^2}{abc}$$

$\Rightarrow \quad 2bc\cos A+ac\cos B+2bc\cos C$

$=a^2+b^2$

$\Rightarrow \quad (b^2+c^2-a^2)+\frac{1}{2}(a^2+c^2-b^2)$

$+(a^2+b^2-c^2)=a^2+b^2$

$\Rightarrow \quad 2b^2-2a^2+c^2+a^2-b^2=0$

$\Rightarrow \quad a^2=b^2+c^2$

ΔABC is a right angled triangle at A

Thus, $\angle A=90°$

31. We have

$$2\left(a\sin^2\left(\frac{C}{2}\right)+c\sin^2\left(\frac{A}{2}\right)\right)$$

$$=\left(2a\sin^2\left(\frac{C}{2}\right)+2\,c\sin^2\left(\frac{A}{2}\right)\right)$$

$$=a(1-\cos C)+c(1-\cos A)$$

$$=a+c-(a\cos C+c\cos A)$$

$$=(a+c-b)$$

32. We have

$$2\left(b\cos^2\left(\frac{C}{2}\right)+c\cos^2\left(\frac{B}{2}\right)\right)$$

$$=\left(2b\cos^2\left(\frac{C}{2}\right)+2\,c\cos^2\left(\frac{B}{2}\right)\right)$$

$$=b(1+\cos C)+c(1+\cos B)$$

$$=b+c+(b\cos C+c\cos B)$$

$$=(b+c+a)$$

$$=(a+b+c)$$

33. We have

$(b+c)\cos A+(c+a)\cos B+(a+b)\cos C$

$=(b\cos A+a\cos B)+(c\cos A+a\cos C)$

$+(b\cos C+c\cos B)$

$=(c+b+a)$

$=(a+b+c)$

34. We have

$$\frac{c-a\cos B}{b-a\cos C}=\frac{a\cos B+b\cos A-a\cos B}{c\cos A+a\cos C-a\cos C}$$

$$=\frac{b\cos A}{c\cos A}$$

$$=\frac{b}{c}$$

$$=\frac{k\sin B}{k\sin C}$$

$$=\frac{\sin B}{\sin C}$$

35. We have

$$2\left(a\sin^2\frac{C}{2}+c\sin^2\frac{A}{2}\right)$$

$$=a\left(2\sin^2\left(\frac{C}{2}\right)\right)+c\left(2\sin^2\left(\frac{A}{2}\right)\right)$$

$$=a(1-\cos C)+c(1-\cos A)$$

$$=a+c-(a\cos C+c\cos A)$$

$$=a+c-b$$

36. We have

$$\frac{\cos A}{b\cos C + c\cos B} + \frac{\cos B}{c\cos A + a\cos C} + \frac{\cos C}{a\cos B + b\cos A}$$

$$= \frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c}$$

$$= \left(\frac{b^2+c^2-a^2}{2abc}\right) + \left(\frac{c^2+a^2-b^2}{2abc}\right) + \left(\frac{a^2+b^2-c^2}{2abc}\right)$$

$$= \frac{1}{2abc}\{(b^2+c^2-a^2) + (c^2+a^2-b^2) + (a^2+b^2-c^2)\}$$

$$= \frac{(a^2+b^2+c^2)}{2abc}$$

37. We have

$2(bc\cos A + ca\cos B + ab\cos C)$

$$= 2bc\cos A + 2ca\cos B + 2ab\cos C$$

$$= 2bc\left(\frac{b^2+c^2-a^2}{2bc}\right) + 2ca\left(\frac{a^2+c^2-b^2}{2ac}\right) + 2ab\left(\frac{a^2+b^2-c^2}{2ab}\right)$$

$$= (b^2+c^2-a^2) + (a^2+c^2-b^2) + (a^2+b^2-c^2)$$

$$= (a^2+b^2+c^2)$$

38. As we know that,

$$\tan\left(\frac{B-C}{2}\right) = \left(\frac{b-c}{b+c}\right)\cot\left(\frac{A}{2}\right)$$

$$= \left(\frac{(\sqrt{3}+1)-(\sqrt{3}-1)}{(\sqrt{3}+1)+(\sqrt{3}-1)}\right)\cot\left(\frac{60^\circ}{2}\right)$$

$$= \frac{1}{\sqrt{3}}\cot(30^\circ)$$

$$= \frac{1}{\sqrt{3}} \times \sqrt{3}$$

$$= 1$$

39. As we know that,

$$\tan\left(\frac{B-C}{2}\right) = \left(\frac{b-c}{b+c}\right)\cot\left(\frac{A}{2}\right)$$

$$\Rightarrow \tan\left(\frac{90^\circ}{2}\right) = \left(\frac{\sqrt{3}-1}{\sqrt{3}+1}\right)\cot\left(\frac{A}{2}\right)$$

$$\Rightarrow \tan(45^\circ) = (2-\sqrt{3})\cot\left(\frac{A}{2}\right)$$

$$\Rightarrow \cot\left(\frac{A}{2}\right) = \frac{1}{(2-\sqrt{3})} = (2+\sqrt{3})$$

$$\Rightarrow \cot\left(\frac{A}{2}\right) = \cot(15^\circ)$$

$$\Rightarrow A = 30^\circ$$

Hence, the angle A is 30°.

40. Given,

$$\cos(A-B) = \frac{4}{5}$$

$$\Rightarrow 2\cos^2\left(\frac{A-B}{2}\right) - 1 = \frac{4}{5}$$

$$\Rightarrow 2\cos^2\left(\frac{A-B}{2}\right) = 1 + \frac{4}{5} = \frac{9}{5}$$

$$\Rightarrow \cos^2\left(\frac{A-B}{2}\right) = \frac{9}{10}$$

$$\Rightarrow \cos\left(\frac{A-B}{2}\right) = \frac{3}{\sqrt{10}}$$

$$\Rightarrow \tan\left(\frac{A-B}{2}\right) = \frac{1}{3}$$

$$\Rightarrow \left(\frac{a-b}{a+b}\right)\cot\left(\frac{C}{2}\right) = \frac{1}{3}$$

$$\Rightarrow \left(\frac{6-3}{6+3}\right)\cot\left(\frac{C}{2}\right) = \frac{1}{3}$$

$$\Rightarrow \frac{1}{3}\cot\left(\frac{C}{2}\right) = \frac{1}{3}$$

$$\Rightarrow \cot\left(\frac{C}{2}\right) = 1$$

$$\Rightarrow \frac{C}{2} = \frac{\pi}{4}$$

$$\Rightarrow C = \frac{\pi}{2}$$

Hence, the value of C is $\frac{\pi}{2}$.

43. We have

$$2s = a + b + c = 13 + 14 + 15$$

$$s = \frac{13+14+15}{2} = \frac{42}{2} = 21$$

(i) $$\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$$

$$= \sqrt{\frac{(21-14)(21-15)}{14.15}}$$

$$= \sqrt{\frac{7.6}{14.15}} = \frac{1}{\sqrt{5}}$$

(ii) $$\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$$

$$= \sqrt{\frac{21(21-13)}{14.15}}$$

$$= \sqrt{\frac{21.8}{14.15}}$$

$$= \sqrt{\frac{4}{5}} = \frac{2}{\sqrt{5}}$$

(iii) $\cos A = 2\cos^2\left(\frac{A}{2}\right) - 1$

$= 2\left(\frac{4}{5}\right) - 1$

$= \frac{8-5}{5} = \frac{3}{5}$

44. We have $\cos\left(\frac{A}{2}\right) = \sqrt{\frac{b+c}{2c}}$

$\Rightarrow \sqrt{\frac{s(s-a)}{bc}} = \sqrt{\frac{b+c}{2c}}$

$\Rightarrow \frac{s(s-a)}{bc} = \frac{b+c}{2c}$

$\Rightarrow 2s(2s-2a) = 2b(b+c)$

$\Rightarrow (a+b+c)(b+c-a) = 2b(b+c)$

$\Rightarrow ((b+c)^2 - a^2) = 2b(b+c)$

$\Rightarrow b^2 + c^2 - a^2 = 2b^2$

$\Rightarrow a^2 + b^2 = c^2$

Thus, the triangle ABC is right angled at C.

45. We have

$b\cos^2\left(\frac{C}{2}\right) + c\cos^2\left(\frac{B}{2}\right)$

$= b\left(\frac{s(s-c)}{ab}\right) + c\left(\frac{s(s-b)}{ac}\right)$

$= \frac{s}{a}(s-c+s-b)$

$= \frac{s}{a}(2s-c-b)$

$= \frac{s}{a}(a+b+c-c-b)$

$= s$

46. We have

$bc\cos^2\left(\frac{A}{2}\right) + ca\cos^2\left(\frac{B}{2}\right) + ab\cos^2\left(\frac{C}{2}\right)$

$= bc\left(\frac{s(s-a)}{bc}\right) + ca\left(\frac{s(s-b)}{ca}\right) + ab\left(\frac{s(s-c)}{ab}\right)$

$= s(s-a) + s(s-b) + s(s-c)$

$= s(3s - (a+b+c))$

$= s(3s - 2s)$

$= s \times s$

$= s^2$

47. We have $2ac\sin\left(\frac{A-B+C}{2}\right)$

$= 2ac\sin\left(\frac{A+C-B}{2}\right)$

$= 2ac\sin\left(\frac{\pi - B - B}{2}\right)$

$= 2ac\sin\left(\frac{\pi}{2} - B\right)$

$= 2ac\cos B$

$= 2ac\left(\frac{a^2+c^2-b^2}{2ac}\right)$

$= (a^2 + c^2 - b^2)$

48. We have $\cot\left(\frac{B}{2}\right)\cot\left(\frac{C}{2}\right)$

$= \sqrt{\frac{s(s-b)}{(s-a)(s-c)}} \times \sqrt{\frac{s(s-c)}{(s-a)(s-b)}}$

$= \sqrt{\frac{s(s-b)}{(s-a)(s-c)} \times \frac{s(s-c)}{(s-a)(s-b)}}$

$= \sqrt{\frac{s^2}{(s-a)^2}}$

$= \frac{s}{s-a}$

$= \frac{2s}{2s-2a}$

$= \frac{a+b+c}{a+b+c-2a}$

$= \frac{4a}{4a-2a}$

$= 2$

49. We have $1 - \tan\left(\frac{A}{2}\right)\tan\left(\frac{B}{2}\right)$

$= 1 - \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}\sqrt{\frac{(s-a)(s-c)}{s(s-b)}}$

$= 1 - \sqrt{\frac{(s-c)^2}{s^2}}$

$= 1 - \frac{(s-c)}{s}$

$= \frac{c}{s}$

$= \frac{2c}{2s}$

$= \frac{2c}{(a+b+c)}$

Hence, the result.

50. We have $\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2}$

$= \sqrt{\frac{s(s-a)}{(s-b)(s-c)}} + \sqrt{\frac{s(s-b)}{(s-a)(s-c)}} + \sqrt{\frac{s(s-c)}{(s-a)(s-b)}}$

$= \sqrt{\frac{s^2(s-a)^2}{s(s-a)(s-b)(s-c)}} + \sqrt{\frac{s^2(s-b)^2}{s(s-b)(s-a)(s-c)}}$

$+ \sqrt{\frac{s^2(s-c)^2}{s(s-c)(s-a)(s-b)}}$

$$=\sqrt{\frac{s^2(s-a)^2}{\Delta^2}}+\sqrt{\frac{s^2(s-b)^2}{\Delta^2}}+\sqrt{\frac{s^2(s-c)^2}{\Delta^2}}$$

$$=\frac{s(s-a)}{\Delta}+\frac{s(s-b)}{\Delta}+\frac{s(s-c)}{\Delta}$$

$$=\frac{s}{\Delta}(s-a+s-b+s-c)$$

$$=\frac{s}{\Delta}(3s-(a+b+c))$$

$$=\frac{s}{\Delta}(3s-2s)$$

$$=\frac{s^2}{\Delta}$$

$$=\frac{s^2}{\sqrt{s(s-a)(s-b)(s-c)}}$$

$$=\frac{s}{(s-a)}\times\sqrt{\frac{s(s-a)}{(s-b)(s-c)}}$$

$$=\frac{2s}{(2s-2a)}\times\sqrt{\frac{s(s-a)}{(s-b)(s-c)}}$$

$$=\frac{(a+b+c)}{(b+c-a)}\times\cot\left(\frac{A}{2}\right)$$

51. Given, $\cot\frac{A}{2}, \cot\frac{B}{2}, \cot\frac{C}{2}$ are in AP

$\Rightarrow\quad 2\cot\left(\frac{B}{2}\right)=\cot\left(\frac{A}{2}\right)+\cot\left(\frac{C}{2}\right)$

$\Rightarrow\quad 2\sqrt{\frac{s(s-b)}{(s-a)(s-c)}}=\sqrt{\frac{s(s-a)}{(s-b)(s-c)}}+\sqrt{\frac{s(s-c)}{(s-a)(s-b)}}$

$\Rightarrow\quad 2\sqrt{\frac{(s-b)}{(s-a)(s-c)}}=\sqrt{\frac{(s-a)}{(s-b)(s-c)}}+\sqrt{\frac{(s-c)}{(s-a)(s-b)}}$

$\Rightarrow\quad 2\sqrt{\frac{(s-b)^2}{(s-b)(s-a)(s-c)}}=\sqrt{\frac{(s-a)^2}{(s-a)(s-b)(s-c)}}+\sqrt{\frac{(s-c)^2}{(s-a)(s-b)(s-c)}}$

$\Rightarrow\quad 2(s-b)=(s-a)+(s-c)$

$\Rightarrow\quad 2(s-b)=(2s-a-c)$

$\Rightarrow\quad 2b=a+c$

$\Rightarrow\quad a, b, c\in$ AP

52. We have $c(a+b)\cos\frac{B}{2}=b(a+c)\cos\frac{C}{2}$

$\Rightarrow\quad c(a+b)\times\sqrt{\frac{s(s-b)}{ac}}=b(a+c)\times\sqrt{\frac{s(s-c)}{ab}}$

$\Rightarrow\quad c(a+b)\times\sqrt{\frac{(s-b)}{c}}=b(a+c)\times\sqrt{\frac{(s-c)}{b}}$

$\Rightarrow\quad c^2(a+b)^2\times\frac{(s-b)}{c}=b^2(a+c)^2\times\frac{(s-c)}{b}$

$\Rightarrow\quad c(a+b)^2\times(s-b)=b(a+c)^2\times(s-c)$

$\Rightarrow\quad c(a+b)^2\times(s-b)=b(a+c)^2\times(s-c)$

$\Rightarrow\quad c(a+b)^2\times(a+c+b)=b(a+c)^2\times(a+b-c)$

$\Rightarrow\quad \frac{(a+c-b)}{b(a+c)^2}=\frac{(a+b-c)}{c(a+b)^2}$

$\Rightarrow\quad \frac{1}{b(a+c)}-\frac{1}{(a+c)^2}=\frac{1}{c(a+b)}-\frac{1}{(a+b)^2}$

$\Rightarrow\quad \frac{1}{b(a+c)}-\frac{1}{c(a+b)}=\frac{1}{(a+c)^2}-\frac{1}{(a+b)^2}$

$\Rightarrow\quad \frac{ac+c^2-ab-b^2}{bc(a+b)(a+c)}=\frac{(a+b)^2-(a+c)^2}{(a+c)^2(a+b)^2}$

$\Rightarrow\quad \frac{a(c-b)+(c^2-b^2)}{bc}=\frac{2a(b-c)+(b^2-c^2)}{(a+c)(a+b)}$

$\Rightarrow\quad \frac{(c-b)(a+b+c)}{bc}=\frac{(b-c)(2a+b+c)}{(a+c)(a+b)}$

$\Rightarrow\quad (c-b)\left(\frac{(a+b+c)}{bc}+\frac{(2a+b+c)}{(a+c)(a+b)}\right)=0$

$\Rightarrow\quad (c-b)=0$

$$\left(\because\left(\frac{(a+b+c)}{bc}+\frac{(2a+b+c)}{(a+c)(a+b)}\right)\neq 0\right)$$

$\Rightarrow\quad b=c$

$\Rightarrow\quad \Delta$ is isosceles

55. $\Delta ABC=\frac{1}{2}\times\sqrt{2}\times\sqrt{3}=\sqrt{\frac{3}{2}}$ s.u.

56. We have

$$\frac{a^2-b^2}{2}\times\frac{\sin A\sin B}{\sin(A-B)}$$

$$=\frac{k^2(\sin^2 A-\sin^2 B)}{2}\times\frac{\sin A\cdot\sin B}{\sin(A-B)}$$

$$=\frac{k^2\times\sin(A+B)\times\sin(A-B)}{2}\times\frac{\sin A\cdot\sin B}{\sin(A-B)}$$

$$=\frac{k^2\times\sin(A+B)\times\sin A\cdot\sin B}{2}$$

$$=\frac{k^2\times\sin(\pi-C)\times\sin A\cdot\sin B}{2}$$