Q. For any positive integer n, define

$f_{\mathrm{n}}:(0, \infty) \rightarrow \square$ as

$f_{\mathrm{n}}(\mathrm{x})=\sum_{\mathrm{j}=1}^{\mathrm{n}} \tan ^{-1}\left(\frac{1}{1+(\mathrm{x}+\mathrm{j})(\mathrm{x}+\mathrm{j}-1)}\right)$ for all $\mathrm{x} \in(0, \infty)$ (Here, the inverse trigonometric function $\tan ^{-1} \mathrm{x}$ assume values in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.) Then, which of the following statement(s) is (are) TRUE? (A) $\sum_{j=1}^{5} \tan ^{2}\left(f_{j}(0)\right)=55$ (B) $\sum_{j=1}^{10}\left(1+f_{j}^{\prime}(0)\right) \sec ^{2}\left(f_{j}(0)\right)=10$ (C) For any fixed positive integer $n, \lim _{x \rightarrow \infty} \tan \left(f_{\mathrm{n}}(x)\right)=\frac{1}{n}$ (D) For any fixed positive integer $n, \operatorname{limsec}_{x \rightarrow \infty} \mathrm{ec}^{2}\left(f_{\mathrm{n}}(x)\right)=1$

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Sol. (D)

$$f_n(x) = \sum_{j=1}^{n} \tan^{-1}\left(\frac{(x+j)-(x+j-1)}{1+(x+j)(x+j-1)}\right)$$

$$f_n(x) = \sum_{j=1}^{n} [\tan^{-1}(x+j) - \tan^{-1}(x+j-1)]$$

$$f_n(x) = \tan^{-1}(x+n) - \tan^{-1}x$$

$$\therefore \quad \tan(f_n(x)) = \tan[\tan^{-1}(x+n) - \tan^{-1}x]$$

$$\tan(f_n(x)) = \frac{(x+n)-x}{1+x(x+n)}$$

$$\tan(f_n(x)) = \frac{n}{1+x^2+nx}$$

$$\therefore \quad \sec^2(f_n(x)) = 1 + \tan^2(f_n(x))$$

$$\sec^2(f_n(x)) = 1+\left(\frac{n}{1+x^2+nx}\right)^2$$

$$\lim_{x\to\infty} \sec^2(f_n(x)) = \lim_{x\to\infty} 1+\left(\frac{n}{1+x^2+nx}\right)^2 = 1$$