4.5 Magnetic Field due to a Current Element, Biot-Savart Law

All magnetic fields that we know are due to currents (or moving charges) and due to intrinsic magnetic moments of particles. Here, we shall study the relation between current and the magnetic field it produces. It is given by the Biot-Savart's law. Figure 4.9 shows a finite conductor XY carrying current I. Consider an infinitesimal element dl of the conductor. The magnetic field d**B** due to this element is to be determined at a point P which is at a distance r from it. Let θ be the angle between d$\mathbf{l}$ and the displacement vector **r**. According to Biot-Savart's law, the magnitude of the magnetic field d**B** is proportional to the current I, the element length $|d\mathbf{l}|$, and inversely proportional to the square of the distance r. Its direction* is perpendicular to the plane containing d$\mathbf{l}$ and **r** . Thus, in vector notation,

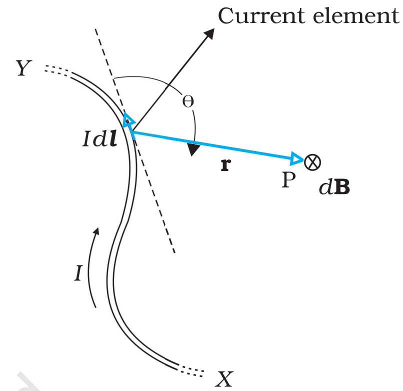

FIGURE 4.9 Illustration of the Biot-Savart law. The current element I d$\mathbf{l}$ produces a field d**B** at a distance r. The ⊗ sign indicates that the field is perpendicular to the plane of this page and directed into it.

$$d\mathbf{B} \propto \frac{I\,d\mathbf{l}\times\mathbf{r}}{r^3}$$

$$= \frac{\mu_0}{4\pi}\frac{I\,d\mathbf{l}\times\mathbf{r}}{r^3} \qquad [4.11(a)]$$

where $\mu_0/4\pi$ is a constant of proportionality. The above expression holds when the medium is vacuum.

The magnitude of this field is,

$$|d\mathbf{B}| = \frac{\mu_0}{4\pi}\frac{I\,dl\sin\theta}{r^2} \qquad [4.11(b)]$$

where we have used the property of cross-product. Equation [4.11 (a)] constitutes our basic equation for the magnetic field. The proportionality constant in SI units has the exact value,

$$\frac{\mu_0}{4\pi} = 10^{-7}\,\mathrm{Tm/A} \qquad [4.11(c)]$$

We call μ_0 the *permeability* of free space (or vacuum).

The Biot-Savart law for the magnetic field has certain similarities, as well as, differences with the Coulomb's law for the electrostatic field. Some of these are:

(i) Both are long range, since both depend inversely on the square of distance from the source to the point of interest. The principle of superposition applies to both fields. [In this connection, note that the magnetic field is *linear* in the *source* $I\,d\mathbf{l}$ just as the electrostatic field is linear in its source: the electric charge.]

(ii) The electrostatic field is produced by a scalar source, namely, the electric charge. The magnetic field is produced by a vector source $I\,d\mathbf{l}$.

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* The sense of $d\mathbf{l}\times\mathbf{r}$ is also given by the *Right Hand Screw rule* : Look at the plane containing vectors d$\mathbf{l}$ and **r**. Imagine moving from the first vector towards second vector. If the movement is anticlockwise, the resultant is towards you. If it is clockwise, the resultant is away from you.

(iii) The electrostatic field is along the displacement vector joining the source and the field point. The magnetic field is perpendicular to the plane containing the displacement vector **r** and the current element $I\,d\boldsymbol{l}$.

(iv) There is an angle dependence in the Biot-Savart law which is not present in the electrostatic case. In Fig. 4.9, the magnetic field at any point in the direction of $d\boldsymbol{l}$ (the dashed line) is zero. Along this line, $\theta = 0$, $\sin\theta = 0$ and from Eq. [4.11(a)], $|d\mathbf{B}| = 0$.

There is an interesting relation between ε_0, the permittivity of free space; μ_0, the permeability of free space; and c, the speed of light in vacuum:

$$\varepsilon_0\mu_0 = (4\pi\varepsilon_0)\frac{\mu_0}{4\pi} = \frac{1}{9\times10^9}(10^{-7}) = \frac{1}{(3\times10^8)^2} = \frac{1}{c^2}$$

We will discuss this connection further in Chapter 8 on the electromagnetic waves. Since the speed of light in vacuum is constant, the product $\mu_0\varepsilon_0$ is fixed in magnitude. Choosing the value of either ε_0 or μ_0, fixes the value of the other. In SI units, μ_0 is fixed to be equal to $4\pi \times 10^{-7}$ in magnitude.

Example 4.5 An element $\Delta\boldsymbol{l} = \Delta x\,\hat{\mathbf{i}}$ is placed at the origin and carries a large current I = 10 A (Fig. 4.10). What is the magnetic field on the y-axis at a distance of 0.5 m. Δx = 1 cm.

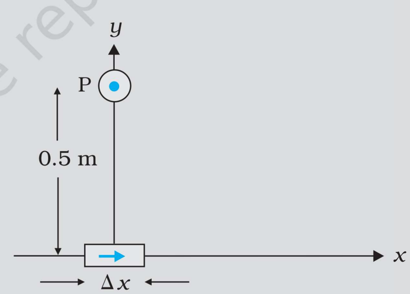

FIGURE 4.10

Solution

$$|d\mathbf{B}| = \frac{\mu_0}{4\pi}\frac{I\,dl\sin\theta}{r^2} \quad \text{[using Eq. (4.11)]}$$

$dl = \Delta x = 10^{-2}$ m, I = 10 A, r = 0.5 m = y, $\mu_0/4\pi = 10^{-7}\,\dfrac{\text{T m}}{\text{A}}$

$\theta = 90°$; $\sin\theta = 1$

$$|d\mathbf{B}| = \frac{10^{-7}\times10\times10^{-2}}{25\times10^{-2}} = 4\times10^{-8}\ \text{T}$$

The direction of the field is in the +z-direction. This is so since,

$$d\boldsymbol{l}\times\mathbf{r} = \Delta x\,\hat{\mathbf{i}}\times y\,\hat{\mathbf{j}} = y\,\Delta x\left(\hat{\mathbf{i}}\times\hat{\mathbf{j}}\right) = y\,\Delta x\,\hat{\mathbf{k}}$$

We remind you of the following cyclic property of cross-products,

$$\hat{\mathbf{i}}\times\hat{\mathbf{j}} = \hat{\mathbf{k}};\ \hat{\mathbf{j}}\times\hat{\mathbf{k}} = \hat{\mathbf{i}};\ \hat{\mathbf{k}}\times\hat{\mathbf{i}} = \hat{\mathbf{j}}$$

Note that the field is small in magnitude.

EXAMPLE 4.5

In the next section, we shall use the Biot-Savart law to calculate the magnetic field due to a circular loop.

4.6 Magnetic Field on the Axis of a Circular Current Loop

In this section, we shall evaluate the magnetic field due to a circular coil along its axis. The evaluation entails summing up the effect of infinitesimal current elements ($I\,d\boldsymbol{l}$) mentioned in the previous section. We assume that the current I is steady and that the evaluation is carried out in free space (i.e., vacuum).

Figure 4.11 depicts a circular loop carrying a steady current I. The loop is placed in the y-z plane with its centre at the origin O and has a radius R. The x-axis is the axis of the loop. We wish to calculate the magnetic field at the point P on this axis. Let x be the distance of P from the centre O of the loop.

Consider a conducting element $d\boldsymbol{l}$ of the loop. This is shown in Fig. 4.11. The magnitude dB of the magnetic field due to $d\boldsymbol{l}$ is given by the Biot-Savart law [Eq. 4.11(a)],

$$dB = \frac{\mu_0}{4\pi}\frac{I\left|d\boldsymbol{l}\times\mathbf{r}\right|}{r^3} \qquad (4.12)$$

Now $r^2 = x^2 + R^2$. Further, any element of the loop will be perpendicular to the displacement vector from the element to the axial point. For example, the element $d\boldsymbol{l}$ in Fig. 4.11 is in the y-z plane, whereas, the displacement vector $\mathbf{r}$ from $d\boldsymbol{l}$ to the axial point P is in the x-y plane. Hence $|d\boldsymbol{l}\times\mathbf{r}| = r\,dl$. Thus,

$$dB = \frac{\mu_0}{4\pi}\frac{I dl}{\left(x^2+R^2\right)} \qquad (4.13)$$

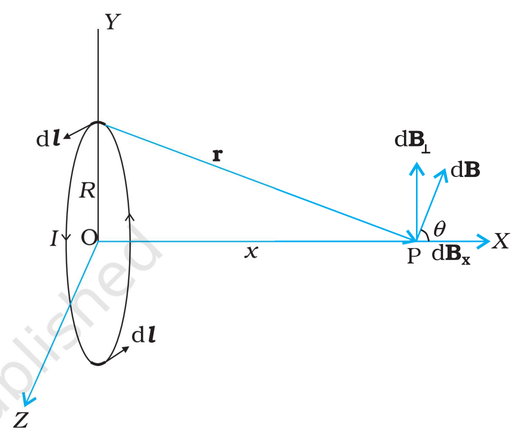

FIGURE 4.11 Magnetic field on the axis of a current carrying circular loop of radius R. Shown are the magnetic field $d\mathbf{B}$ (due to a line element $d\boldsymbol{l}$) and its components along and perpendicular to the axis.

The direction of $d\mathbf{B}$ is shown in Fig. 4.11. It is perpendicular to the plane formed by $d\boldsymbol{l}$ and $\mathbf{r}$. It has an x-component $d\mathbf{B}_x$ and a component perpendicular to x-axis, $d\mathbf{B}_\perp$. When the components perpendicular to the x-axis are summed over, they cancel out and we obtain a null result. For example, the $d\mathbf{B}_\perp$ component due to $d\boldsymbol{l}$ is cancelled by the contribution due to the diametrically opposite $d\boldsymbol{l}$ element, shown in Fig. 4.11. Thus, only the x-component survives. The net contribution along x-direction can be obtained by integrating $dB_x = dB\cos\theta$ over the loop. For Fig. 4.11,

$$\cos\theta = \frac{R}{(x^2+R^2)^{1/2}} \qquad (4.14)$$

From Eqs. (4.13) and (4.14),

$$dB_x = \frac{\mu_0 I dl}{4\pi}\frac{R}{\left(x^2+R^2\right)^{3/2}}$$

The summation of elements dl over the loop yields $2\pi R$, the circumference of the loop. Thus, the magnetic field at P due to entire circular loop is

$$\mathbf{B} = B_x \hat{\mathbf{i}} = \frac{\mu_0 I R^2}{2\left(x^2 + R^2\right)^{3/2}} \hat{\mathbf{i}} \tag{4.15}$$

As a special case of the above result, we may obtain the field at the centre of the loop. Here $x = 0$, and we obtain,

$$\mathbf{B}_0 = \frac{\mu_0 I}{2R} \hat{\mathbf{i}} \tag{4.16}$$

The magnetic field lines due to a circular wire form closed loops and are shown in Fig. 4.12. The direction of the magnetic field is given by (another) *right-hand thumb rule* stated below:

Curl the palm of your right hand around the circular wire with the fingers pointing in the direction of the current. The right-hand thumb gives the direction of the magnetic field.

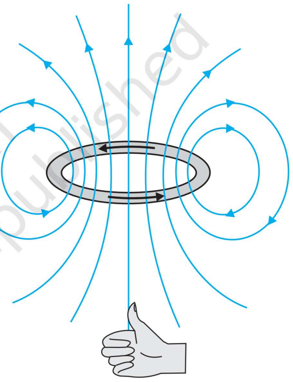

FIGURE 4.12 The magnetic field lines for a current loop. The direction of the field is given by the right-hand thumb rule described in the text. The upper side of the loop may be thought of as the north pole and the lower side as the south pole of a magnet.

EXAMPLE 4.6

Example 4.6 A straight wire carrying a current of 12 A is bent into a semi-circular arc of radius 2.0 cm as shown in Fig. 4.13(a). Consider the magnetic field **B** at the centre of the arc. (a) What is the magnetic field due to the straight segments? (b) In what way the contribution to **B** from the semicircle differs from that of a circular loop and in what way does it resemble? (c) Would your answer be different if the wire were bent into a semi-circular arc of the same radius but in the opposite way as shown in Fig. 4.13(b)?

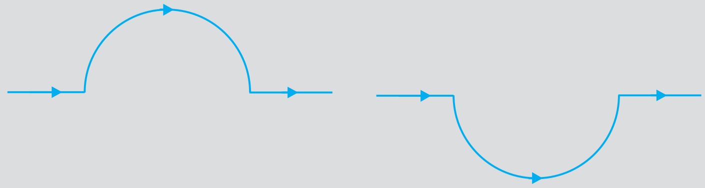

FIGURE 4.13