Find the minimum value of $\log_x a + \log_a x$, $x \geq a > 0$

SOLUTION :

$$0 < a \leq x$$

$$\log_a^x + \log_x^a$$

$$l = \log_a^x + \frac{1}{\log_a^x} \qquad \left(\because \log_b^a = \frac{1}{\log_a^b} \right)$$

let $z = \log_a^x$,

since , $x \geq a > 0$

$$\Rightarrow \log_a^x > 0$$

$$\Rightarrow z > 0$$

$$l = z + \frac{1}{z}$$

now apply $AM \geqslant GM$.

$$\frac{z + \frac{1}{z}}{2} \geqslant \sqrt{z \cdot \frac{1}{z}}$$

$$z + \frac{1}{z} \geqslant 2$$

minimum value of $\log_a^x + \log_x^a$ is 2