Let a >0 , b >0 , c >0 and find the minimum value of $\frac{b+c}{a} + \frac{a+c}{b} + \frac{a+b}{c}$

SOLUTION :

$$\left(\frac{b+c}{a}\right) + \left(\frac{c+a}{b}\right) + \left(\frac{a+b}{c}\right)$$

We can't apply AM , GM or AM , HM or GM , HM inequalities directly , so, we need to manipulate the given expression a bit

Now add and subtract 1 to each term

$$\left(\frac{b+c}{a}+1\right) + \left(\frac{c+a}{b}+b\right) + \left(\frac{a+b}{c}+1\right) - 3$$

$$\left(\frac{a+b+c}{a}\right) + \left(\frac{a+b\,c}{b}\right) + \left(\frac{a+b+c}{c}\right) - 3$$

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) - 3$$

We can't use AM , GM inequality to the above given expression so

Now apply AM $\geq$ HM for a , b , c

We know that,

$$AM \geq HM$$

$$\frac{a+b+c}{3} \geq \frac{3}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}$$

$$\Rightarrow (a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \geq 9$$

So , the minimum value will be

$$9 - 3 = 6$$