

$$22. \cot x \cot 2x - \cot 2x \cot 3x - \cot 3x \cot x = 1$$

$$23. \tan 4x = \frac{4 \tan x (1 - \tan^2 x)}{1 - 6 \tan^2 x + \tan^4 x}$$

$$24. \cos 4x = 1 - 8 \sin^2 x \cos^2 x$$

$$25. \cos 6x = 32 \cos^6 x - 48 \cos^4 x + 18 \cos^2 x - 1$$

### 3.5 Trigonometric Equations

Equations involving trigonometric functions of a variable are called *trigonometric equations*. In this Section, we shall find the solutions of such equations. We have already learnt that the values of  $\sin x$  and  $\cos x$  repeat after an interval of  $2\pi$  and the values of  $\tan x$  repeat after an interval of  $\pi$ . The solutions of a trigonometric equation for which  $0 \leq x < 2\pi$  are called *principal solutions*. The expression involving integer 'n' which gives all solutions of a trigonometric equation is called the *general solution*. We shall use 'Z' to denote the set of integers.

The following examples will be helpful in solving trigonometric equations:

**Example 18** Find the principal solutions of the equation  $\sin x = \frac{\sqrt{3}}{2}$ .

**Solution** We know that,  $\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$  and  $\sin \frac{2\pi}{3} = \sin \left( \pi - \frac{\pi}{3} \right) = \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$ .

Therefore, principal solutions are  $x = \frac{\pi}{3}$  and  $\frac{2\pi}{3}$ .

**Example 19** Find the principal solutions of the equation  $\tan x = -\frac{1}{\sqrt{3}}$ .

**Solution** We know that,  $\tan \frac{\pi}{6} = \frac{1}{\sqrt{3}}$ . Thus,  $\tan \left( \pi - \frac{\pi}{6} \right) = -\tan \frac{\pi}{6} = -\frac{1}{\sqrt{3}}$

and  $\tan \left( 2\pi - \frac{\pi}{6} \right) = -\tan \frac{\pi}{6} = -\frac{1}{\sqrt{3}}$

Thus  $\tan \frac{5\pi}{6} = \tan \frac{11\pi}{6} = -\frac{1}{\sqrt{3}}$ .

Therefore, principal solutions are  $\frac{5\pi}{6}$  and  $\frac{11\pi}{6}$ .

We will now find the general solutions of trigonometric equations. We have already

seen that:

$$\sin x = 0 \text{ gives } x = n\pi, \text{ where } n \in \mathbf{Z}$$

$$\cos x = 0 \text{ gives } x = (2n + 1)\frac{\pi}{2}, \text{ where } n \in \mathbf{Z}.$$

We shall now prove the following results:

**Theorem 1** For any real numbers  $x$  and  $y$ ,

$$\sin x = \sin y \text{ implies } x = n\pi + (-1)^n y, \text{ where } n \in \mathbf{Z}$$

**Proof** If  $\sin x = \sin y$ , then

$$\sin x - \sin y = 0 \text{ or } 2\cos \frac{x+y}{2} \sin \frac{x-y}{2} = 0$$

which gives  $\cos \frac{x+y}{2} = 0$  or  $\sin \frac{x-y}{2} = 0$

Therefore  $\frac{x+y}{2} = (2n+1)\frac{\pi}{2}$  or  $\frac{x-y}{2} = n\pi$ , where  $n \in \mathbf{Z}$

i.e.  $x = (2n+1)\pi - y$  or  $x = 2n\pi + y$ , where  $n \in \mathbf{Z}$

Hence  $x = (2n+1)\pi + (-1)^{2n+1}y$  or  $x = 2n\pi + (-1)^{2n}y$ , where  $n \in \mathbf{Z}$ .

Combining these two results, we get

$$x = n\pi + (-1)^n y, \text{ where } n \in \mathbf{Z}.$$

**Theorem 2** For any real numbers  $x$  and  $y$ ,  $\cos x = \cos y$ , implies  $x = 2n\pi \pm y$ , where  $n \in \mathbf{Z}$

**Proof** If  $\cos x = \cos y$ , then

$$\cos x - \cos y = 0 \text{ i.e., } -2\sin \frac{x+y}{2} \sin \frac{x-y}{2} = 0$$

Thus  $\sin \frac{x+y}{2} = 0$  or  $\sin \frac{x-y}{2} = 0$

Therefore  $\frac{x+y}{2} = n\pi$  or  $\frac{x-y}{2} = n\pi$ , where  $n \in \mathbf{Z}$

i.e.  $x = 2n\pi - y$  or  $x = 2n\pi + y$ , where  $n \in \mathbf{Z}$

Hence  $x = 2n\pi \pm y$ , where  $n \in \mathbf{Z}$

**Theorem 3** Prove that if  $x$  and  $y$  are not odd multiple of  $\frac{\pi}{2}$ , then

$$\tan x = \tan y \text{ implies } x = n\pi + y, \text{ where } n \in \mathbf{Z}$$

**Proof** If  $\tan x = \tan y$ , then  $\tan x - \tan y = 0$

or 
$$\frac{\sin x \cos y - \cos x \sin y}{\cos x \cos y} = 0$$

which gives  $\sin(x - y) = 0$  (Why?)

Therefore  $x - y = n\pi$ , i.e.,  $x = n\pi + y$ , where  $n \in \mathbf{Z}$

**Example 20** Find the solution of  $\sin x = -\frac{\sqrt{3}}{2}$ .

**Solution** We have  $\sin x = -\frac{\sqrt{3}}{2} = -\sin \frac{\pi}{3} = \sin\left(\pi + \frac{\pi}{3}\right) = \sin \frac{4\pi}{3}$

Hence  $\sin x = \sin \frac{4\pi}{3}$ , which gives

$$x = n\pi + (-1)^n \frac{4\pi}{3}, \text{ where } n \in \mathbf{Z}.$$

**Note**  $\frac{4\pi}{3}$  is one such value of  $x$  for which  $\sin x = -\frac{\sqrt{3}}{2}$ . One may take any other value of  $x$  for which  $\sin x = -\frac{\sqrt{3}}{2}$ . The solutions obtained will be the same although these may apparently look different.

**Example 21** Solve  $\cos x = \frac{1}{2}$ .

**Solution** We have,  $\cos x = \frac{1}{2} = \cos \frac{\pi}{3}$

Therefore  $x = 2n\pi \pm \frac{\pi}{3}$ , where  $n \in \mathbf{Z}$ .

**Example 22** Solve  $\tan 2x = -\cot\left(x + \frac{\pi}{3}\right)$ .

**Solution** We have,  $\tan 2x = -\cot\left(x + \frac{\pi}{3}\right) = \tan\left(\frac{\pi}{2} + x + \frac{\pi}{3}\right)$

or  $\tan 2x = \tan\left(x + \frac{5\pi}{6}\right)$

Therefore  $2x = n\pi + x + \frac{5\pi}{6}$ , where  $n \in \mathbf{Z}$

or  $x = n\pi + \frac{5\pi}{6}$ , where  $n \in \mathbf{Z}$ .

**Example 23** Solve  $\sin 2x - \sin 4x + \sin 6x = 0$ .

**Solution** The equation can be written as

$$\sin 6x + \sin 2x - \sin 4x = 0$$

or  $2 \sin 4x \cos 2x - \sin 4x = 0$

i.e.  $\sin 4x(2 \cos 2x - 1) = 0$

Therefore  $\sin 4x = 0$  or  $\cos 2x = \frac{1}{2}$

i.e.  $\sin 4x = 0$  or  $\cos 2x = \cos \frac{\pi}{3}$

Hence  $4x = n\pi$  or  $2x = 2n\pi \pm \frac{\pi}{3}$ , where  $n \in \mathbf{Z}$

i.e.  $x = \frac{n\pi}{4}$  or  $x = n\pi \pm \frac{\pi}{6}$ , where  $n \in \mathbf{Z}$ .

**Example 24** Solve  $2 \cos^2 x + 3 \sin x = 0$

**Solution** The equation can be written as

$$2(1 - \sin^2 x) + 3 \sin x = 0$$

or  $2 \sin^2 x - 3 \sin x - 2 = 0$

or  $(2 \sin x + 1)(\sin x - 2) = 0$

Hence  $\sin x = -\frac{1}{2}$  or  $\sin x = 2$

But  $\sin x = 2$  is not possible (Why?)

Therefore  $\sin x = -\frac{1}{2} = \sin \frac{7\pi}{6}$ .

Hence, the solution is given by

$$x = n\pi + (-1)^n \frac{7\pi}{6}, \text{ where } n \in \mathbf{Z}.$$

### EXERCISE 3.4

Find the principal and general solutions of the following equations:

1.  $\tan x = \sqrt{3}$
2.  $\sec x = 2$
3.  $\cot x = -\sqrt{3}$
4.  $\operatorname{cosec} x = -2$

Find the general solution for each of the following equations:

5.  $\cos 4x = \cos 2x$
6.  $\cos 3x + \cos x - \cos 2x = 0$
7.  $\sin 2x + \cos x = 0$
8.  $\sec^2 2x = 1 - \tan 2x$
9.  $\sin x + \sin 3x + \sin 5x = 0$

### Miscellaneous Examples

**Example 25** If  $\sin x = \frac{3}{5}$ ,  $\cos y = -\frac{12}{13}$ , where  $x$  and  $y$  both lie in second quadrant, find the value of  $\sin(x+y)$ .

**Solution** We know that

$$\sin(x+y) = \sin x \cos y + \cos x \sin y \quad \dots (1)$$

$$\text{Now } \cos^2 x = 1 - \sin^2 x = 1 - \frac{9}{25} = \frac{16}{25}$$

$$\text{Therefore } \cos x = \pm \frac{4}{5}$$

Since  $x$  lies in second quadrant,  $\cos x$  is negative.

$$\text{Hence } \cos x = -\frac{4}{5}$$

$$\text{Now } \sin^2 y = 1 - \cos^2 y = 1 - \frac{144}{169} = \frac{25}{169}$$

$$\text{i.e. } \sin y = \pm \frac{5}{13}$$

Since  $y$  lies in second quadrant, hence  $\sin y$  is positive. Therefore,  $\sin y = \frac{5}{13}$ . Substituting the values of  $\sin x$ ,  $\sin y$ ,  $\cos x$  and  $\cos y$  in (1), we get