Finding the maximum and minimum values of a given function, $f(x)$ on a given interval I.

Recall:

Theorem: If $f(x)$ is a cont. function in a closed interval $[a,b]$, then the function attains its maximum and minimum value on the interval $[a,b]$ i.e., there exists points x_0 and $y_0 \in [a,b]$ s.t $f(x_0) \leq f(x) \leq f(y_0) \quad \forall \; x \in [a,b]$

In case, $f(x_0)$ is minimum value of $f(x)$ on the interval $[a,b]$ and it is attained at point $x_0 \in [a,b]$

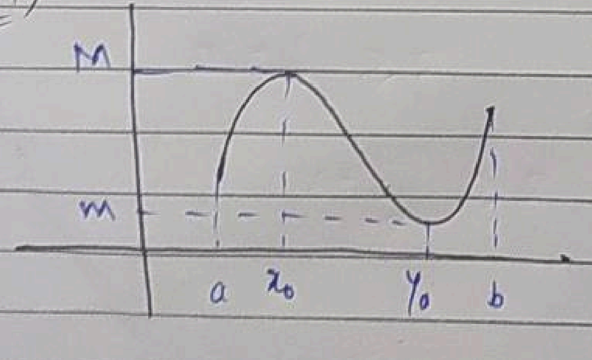
Similarily, $f(y_0)$ is maximum value of $f(x)$ on $[a,b]$ and it is attained at the point $y_0 \in [a,b]$

Note, there can exist more than one point at which the maximum (or minimum) values occur.

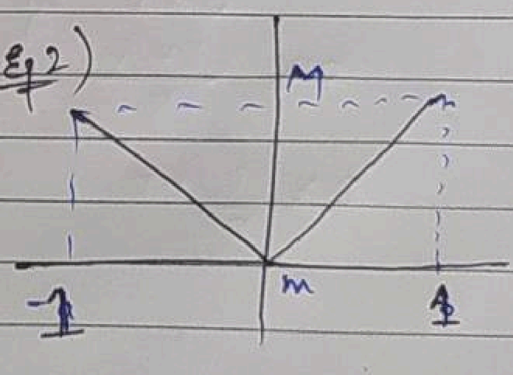
But, there is only one unique $\max^m$ & $\min^m$ value.

Q) Find max. & min. values?

Eg 1)



Eg 2)



Conclusion: The points of absolute maxima & minima of $f(x)$ on $[a,b]$ can be either one of the end points or an interior point.

Definition: For a continous function, on an interval $[a,b]$ the critical points of f are the points where f is not differentiable or where the derivate equals zero.

So, to find the points of maxima & minima and maximum & minimum values.

Step 1: Find all critical points in open interval. (a,b)

Step 2: Find values of $f(x)$ at all critical points and at end points.

Step 3: Find min^m & max^m values among the obtained values.

Example: Find all max^m & min^m values of $f(x) = x^3 - 6x^2 + 9x + 15$ on $[-1,2]$

- Critical pts:

$$f'(x) = 3x^2 - 12x + 9$$

$$f'(x) = 0$$

$$3x^2 - 12x + 9 = 0$$

$$x = 1 \text{ or } x = 3$$

Now, $x = 1$ lies in $[-1, 2]$

Only required critical pt. is 1.

$f(1) = 1 - 6 + 9 + 15 = 19$

$f(-1) = (-1)^3 - 6(-1)^2 + 9(-1) + 15 = -1.$

$f(2) = 2^3 - 6(2)^2 + 9(2) + 15 = 17$

Max^m value $= 19$ at $x = 1.$

Min^m value $= -1$ at $x = -1.$

Q) consider $f(x) = x + \frac{1}{x}$, $x \in \mathbb{R} \setminus \{0\}$

Find max^m & min^m values of $f(x)$ if they exist.

- $f(x) = x + \frac{1}{x}$, $x \neq 0$

$$x^2 + 1 - 2x = (x-1)^2 \geq 0$$

$$x^2 + 1 \geq 2x$$

if $x > 0$

$$\frac{x^2+1}{x} \geq 2$$

$$x + \frac{1}{x} \geq 2$$

if $x < 0$

$$\frac{x^2+1}{x} \leq a$$

$$x + \frac{1}{x} \leq a$$

So, on the interval $(0, \infty)$ $f(x) = x + \frac{1}{x} \geqslant 2$

Also, at $x = 1$, $f(x) = 2$

Hence, $f(x)$ has minimum value 2 attained at $x = 1$ on the interval. $(0, \infty)$

No $\max^m$ value as $x \to +\infty$
$f(x) \to +\infty$