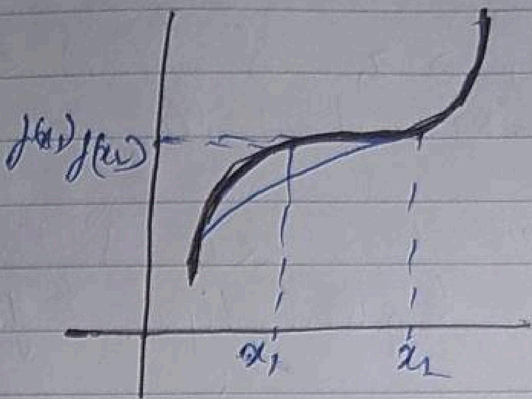
Terminology:- * ↑ → increasing
* ↓ → decreasing.

# Monatonocity

Nature of graph.

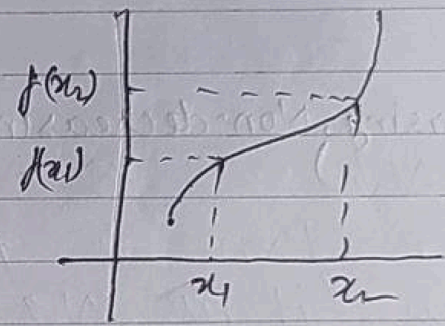
① If $f(x)$ in ↑ $(a,b)$




a.) If $x_2 \ge x_1 \Rightarrow f(x_2) \ge f(x_1)$
then $f(x)$ is ↑ in $a, b$

b) If $f(x)$ is defined &
$f'(x) \ge 0$ then $f(x)$ is ↑

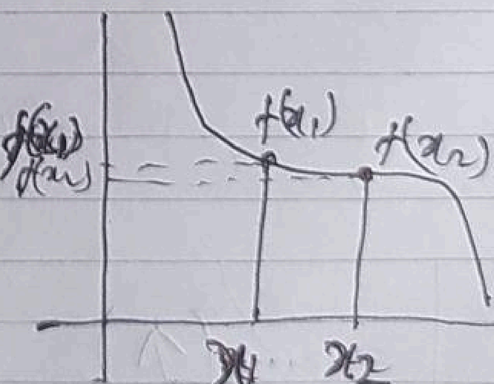
2. If $f(x)$ is strictly ↑ in $(a, b)$




a. If $x_2 > x_1 \Rightarrow f(x_2) > f(x_1)$
then $f(x)$ is
strictly increasing in
$(a, b)$

b. If $f'(x)$ defined $f'(x) \ge 0$ when $f'(x) = 0$
is Psble only for an instant the
also $f^n$ is st. ↑.

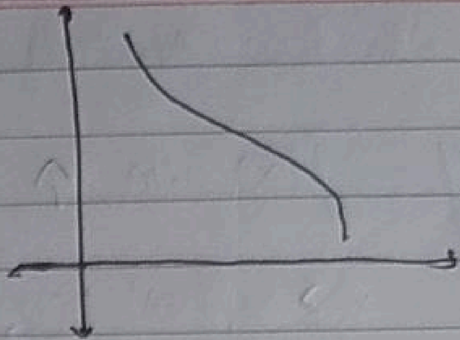
3. If $f(x)$ is decreasing in $(a, b)$




a. $x_1 \le x_2 \Rightarrow f(x_1) \ge f(x_2)$
then $f(x)$ is ↓,

b. If $f'(x) \le 0$ then
$f(x)$ is ↓.

4. If $f(x)$ is str. ↓



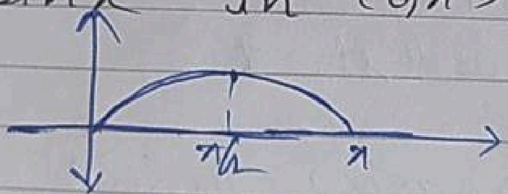
a. $x_1 > x_2 \Rightarrow f(x_1) < f(x_2)$
then $f(x)$ is str. ↓.

b. If $f'(x) \leq 0 \Rightarrow f(x)$ is str ↓.
Equality for an instant.

5. Montonic fxn:
Any fxn is said to be monotonic if it is either ↑ or ↓ in $(a, b)$
$f'(x) \geq 0$ $\quad$ $f'(x) \leq 0$

Non-increasing, Non-decreasing fxn
6. (N↑ N↓) या Non-monotonic fxn:
If a fxn is ↑ & ↓ both in $(a, b)$ then it is N↑ N↓ or non-monotonic fxn

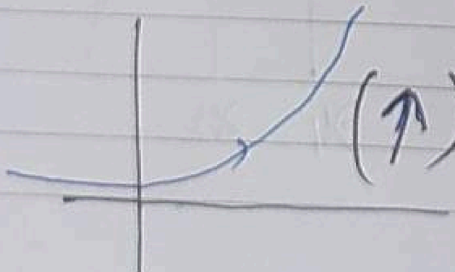
Eg - $\sin x$ in $(0, \pi)$




↑ in $(0, \pi/2)$
↓ in $(\pi/2, \pi)$

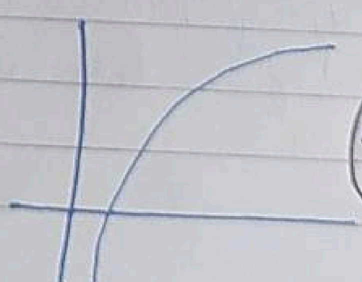
in $(0, \pi)$
non-montonic

Q. 1. $y = e^x$



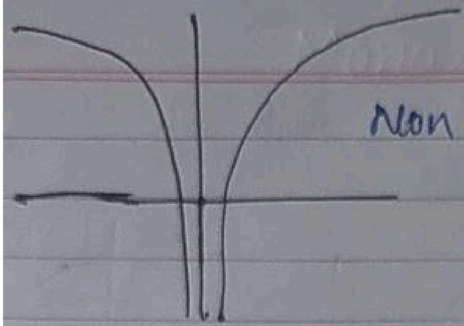
(↑)

2. $y = \ln x$



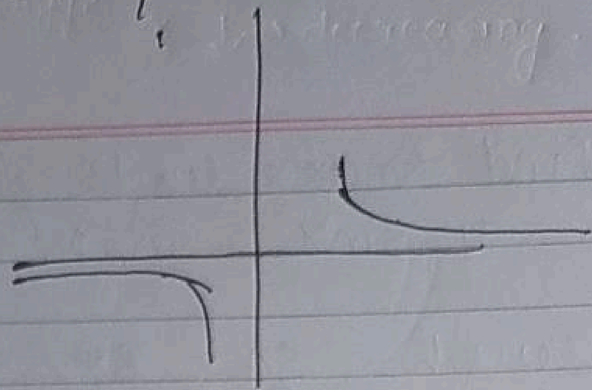
(↑)

③ $y = \ln|x|$



Non montonic

④ $y = \operatorname{cosec}^{-1} x$



7. <u>Montonic Behaviour at $x = a$</u>

$h \to 0$

$f(a-h) < f(a) < f(a+h)$
$f(x)$ is st $\uparrow$ at $x = a$

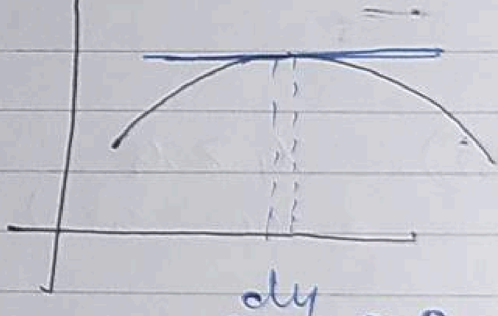
$f(a-h) \le f(a) \le f(a+h)$
$f(x)$ is $\uparrow$ at $x = a$

$f(a-h) > f(a) > f(a+h)$
$f(x)$ is $\downarrow$ at $x = a$

$f(a-h) \ge f(a) \ge f(a+h)$
$f(x)$ is st. $\downarrow$ at $x = a$

8. <u>Stationary Point</u> -

Point where $f'(x) = 0$ is a stationery point.



$\frac{dy}{dx} = 0$

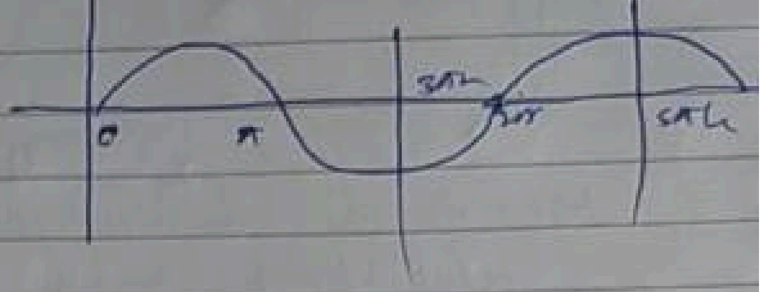
9. <u>Critical Pt.</u>

A) If $f'(x) = 0$ or $f'(x) = $ DNE then it is cr. Pt.

B) Every stationery pt is critical point.

C) At stationery pt. tangent is ∥ to x-axis.

Q. Find max. length of interval in which $y = \sin x$ is ↑.

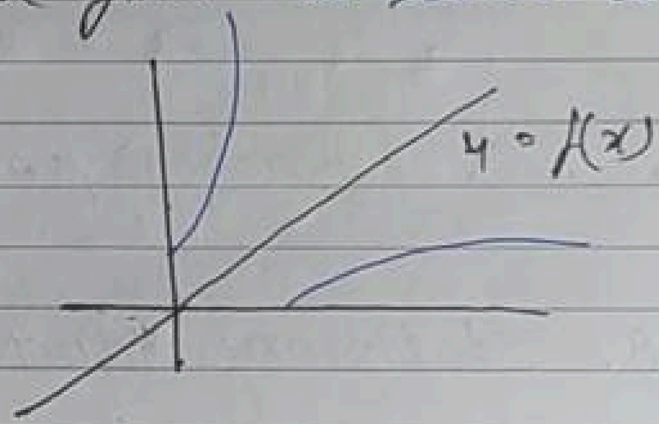



length $= \frac{5\pi}{2} - \frac{3\pi}{2} = \pi$

~~If it is asked~~

10) If $f(x)$ is one-one or $f(x)$ is invertible then $f'(x) > 0$ or $f'(x) < 0$ or $f(x)$ is monotonic.

11) Behaviour of inverse fxn is same as its basic fxn.




Basic fxn ↑
⇒ Inverse fxn ↑

12) $x_1 > x_2 \Rightarrow f(x_1) > f(x_2)$ St. ↑
$x_1 > x_2 \Rightarrow f(x_2) > f(x_1)$ St. ↓

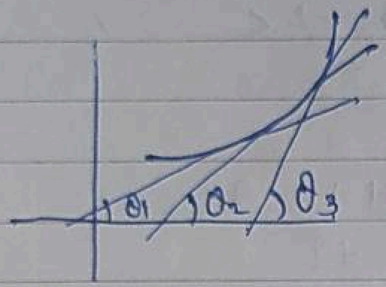
It shows that if ↓ fxn is applied or removed then sign of inequality gets changed.

13) Continuity & differentiability makes no effect in ↑ or ↓ behaviour.

14) If $dy/dx$ is not psbl then use basic method.

15) If $f(x) \uparrow$ then $f'(x) > 0$
$f'(x) \uparrow$ then $f''(x) > 0$

$\frac{d^2y}{dx^2} > 0 \longrightarrow$ concavity up




$f'(x) \uparrow$
$f'(x)$ बढ़ रहा है।
$\Rightarrow$ Slope of curve is $\uparrow$

$\theta$ is increasing. So, $\tan\theta$ is $\uparrow$.

Q. Apply $\sin^{-1}$ & $\cos^{-1}$ . $x-1$ & $x+1$
$x-1 < x+1$
$\sin^{-1} \uparrow, \sin^{-1}(x-1) < \sin^{-1}(x+1)$
$\cos^{-1} \downarrow, \cos^{-1}(x-1) > \cos^{-1}(x+1)$

Q. If $f(x)$ & $g(x)$ 2 fxn's such that if $x_1 > x_2$ ~~gives~~ implies $f(x_1) > f(x_2)$ & $g(x_1) < g(x_2)$ then find interval of $\alpha$ if
$f(g(\alpha^2 - 2\alpha)) > f(g(3\alpha - 4))$

$f \uparrow$ , $g \downarrow$
$g(\alpha^2 - 2\alpha) > g(3\alpha - 4)$
$\alpha^2 - 2\alpha < 3\alpha - 4$
$\alpha^2 - 5\alpha + 4 < 0$
$(\alpha - 1)(\alpha - 4) < 0$
$\alpha \in (1, 4)$

Q. Check if $f'(x) > 0$ & $g'(x) < 0$ & $f(g(x)) > f(g(x+1))$ [T / F]

$f \uparrow$ , $g \downarrow$

$x < x+1$

$g(x) \geq g(x+1)$

$f(g(x)) \geq f(g(x+1))$ <u>True</u>

Q. If $f \times n$ is defined $[0, \infty) \to R$

$$f(x) = \frac{e^{x^2} - e^{-x^2}}{e^{x^2} + e^{-x^2}}$$

$$f(x) = \frac{e^{2x^2} - 1}{e^{2x^2} + 1} = \frac{e^{2x^2} + 1 - 2}{e^{2x^2} + 1}$$

$$f(x) = 1 - \frac{2}{e^{2x^2} + 1}$$

$$f'(x) = -2 \times \left(\frac{-1}{e^{2x^2} + 1}\right) e^{2x^2} \cdot 4x$$

$$f'(x) = \frac{8xe^{2x^2}}{e^{2x^2} + 1}$$

for $x > 0$ , $f'(x) > 0$

<u>Aliter</u> $f(x) = 1 - \frac{2}{e^{2x^2} + 1}$

for $x > 0$, $2x^2 \uparrow$

$e^{2x^2} \uparrow$

$e^{2x^2} + 1 \uparrow$

$\frac{1}{e^{2x^2} + 1} \downarrow$

$-\frac{2}{e^{2x^2} + 1} \uparrow$

$1 - \frac{2}{e^{2x^2} + 1} \uparrow$

# Finding Interval Based Q's

Q. $f(x) = x - e^x + \tan^{-1}\left(\frac{2\pi}{7}\right)$ is ↑ then find $x$?

$$f'(x) = 1 - e^x + 0 \geq 0$$
$$1 \geq e^x$$
$$e^0 \geq e^x$$
$$x \leq 0 \Rightarrow x \in (-\infty, 0]$$

Q. $f(x) = x + \frac{1}{x}$ ↑ in the interval?

$$f'(x) = 1 - \frac{1}{x^2} \geq 0$$
$$1 \geq 1/x^2$$
$$x^2 \geq 1$$
$$|x| \geq 1$$
$$x \in (-\infty, 1] \cup [1, \infty)$$

Q. $f(x) = x + \frac{4}{x^2}$ ↓ then int?

$$f'(x) = 1 - \frac{8}{x^3} \leq 0$$
$$x^3 - 8 \leq 0$$ → +ive
$$(x-2)(x^2 - 2x - 4)/x^3 \leq 0$$
$$(x-2)/x^3 \leq 0$$
$$\frac{x-2}{x} \leq 0$$

$+ \quad 0 \quad - \quad 2 \quad +$

$$x \in [0, 2]$$

Q. $f(x) = \int e^x (x-1)(x-2)\, dx$ is ↓ in ?

$$f'(x) = \overbrace{e^x}^{+ve} (x-1)(x-2) \le 0$$

$$(x-1)(x-2) \le 0$$

$$x \in [1, 2]$$

Q. If $f(x) = kx^3 - 9x^2 + 9x + 3$ ↑ then $k \in$ ?

$$f'(x) = 3kx^2 - 18x + 9 \geq 0$$

$$kx^2 - 6x + 3 \geq 0$$

$k > 0$ | $D \le 0$

$36 - 4 \times 3k \le 0$

$k \geq 3$

$$k \in [3, \infty)$$

Q. If $h(x) = f(x) - (f(x))^2 + (f(x))^3$ then relate $h(x)$ & $f(x)$ monotonic behaviour.

$$h'(x) = f'(x) - 2f(x) f'(x) + 3(f(x))^2 f'(x)$$

$$= f'(x) \left( 3(f(x))^2 - 2f(x) + 1 \right)$$

QE, $D = 4 - 12 = -8 < 0$

$h'(x)$ depend upon $f'(x)$

If $f'(x) > 0 \rightarrow h'(x) > 0$

$f'(x) < 0 \rightarrow h'(x) < 0$

$\Rightarrow$ If $f(x)$ is ↑ then $h(x)$ is ↑ & vice versa.

Q $f(x) = \cos\left(\frac{\pi}{x}\right)$ is ↑ then interval.

+ve

$$f'(x) = -\sin\left(\frac{\pi}{x}\right) \times \frac{-\pi}{x^2} = \frac{\pi}{x^2}\sin\left(\frac{\pi}{x}\right)$$

$f'(x)$ depend upon $\sin(\pi/x)$

$f'(x) > 0$

$\sin\left(\frac{\pi}{x}\right) > 0$

$$2n\cancel{\pi} < \frac{\cancel{\pi}}{x} < (2n+1)\cancel{\pi}$$

$$\frac{1}{2n+1} < x < \frac{1}{2n}$$

Q. If $f(x) = \sin^4 x + \cos^4 x$ is ↑ then interval?

$f(x) = 1 - 2\sin^2 x \cos^2 x$

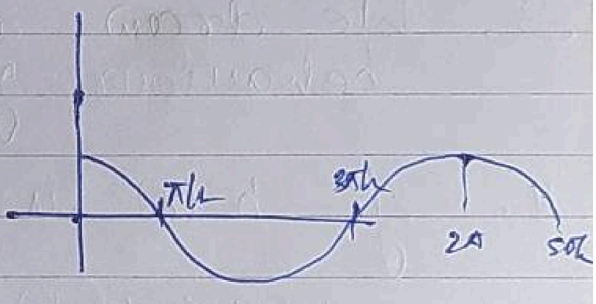
$f(x) = 1 - \frac{\sin 2x}{2}$

$f'(x) = -\frac{\cos 2x}{2}$

$f'(x) > 0$

$-\cos 2x > 0$

$\cos 2x < 0$




$$\frac{4n+1}{2} < 2x < \frac{4n+3}{2}$$

$$\frac{4n+1}{4} < x < \frac{4n+3}{4}$$

Q. If $f(x) = \tan^{-1}(\sin x + \cos x)$ ↑ then $x \in$?

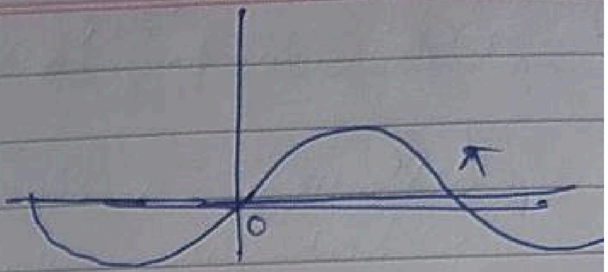
$f(x) = \tan^{-1}\left[\sqrt{2}\sin\left(\frac{\pi}{4} + x\right)\right]$

$f'(x) > 0$

$\tan^{-1}\left(\sqrt{2}\sin\left(\frac{\pi}{4} + x\right)\right) \geq 0$

$$\sqrt{2}\sin\left(\frac{\pi}{4}+x\right) > 0$$

$$\sin\left(\frac{\pi}{4}+x\right) > 0$$




$$2n\pi < \frac{\pi}{4}+x < (2n+1)\pi$$

$$2n\pi - \frac{\pi}{4} < x < (2n+1)\pi - \frac{\pi}{4}$$

## Inequality Based Q's

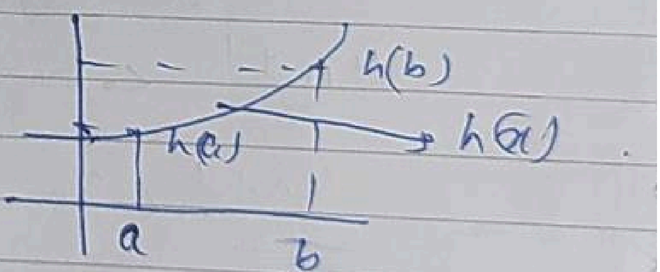
1. If we need to prove $f(x) > g(x)$ then, we take $h(x) = f(x) - g(x)$

2. Now, we check $h(x)$ is ↑ or ↓ in given interval.

3. We draw a graph to understand behaviour of $h(x)$

4. If $h(x)$ is ↑ in $(a, b)$

$\left.\begin{array}{l} h(x) < h(b) \\ h(x) > h(a) \end{array}\right\}$ we use this.




Q. Check $\sin x > x$ in $(0, \pi/2)$ or not?

Let $h(x) = \sin x - x$

$h'(x) = \cos x - 1 \leq 0$

$h'(x)$ ↓ $f x n$.

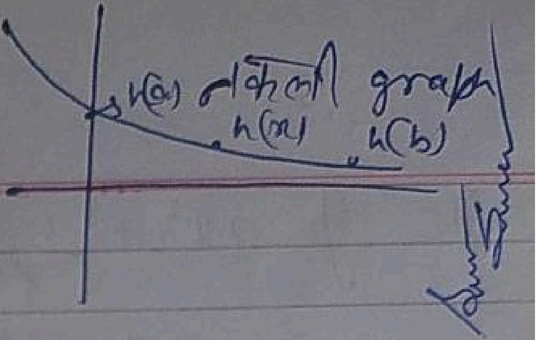
$h(x) < h(0)$

$\sin x - x < \sin 0 - 0$

$\sin x - x < 0$

$\sin x < x$

So, statement is false.




Q. $1 + x \ln(x + \sqrt{1+x^2}) \geq \sqrt{1+x^2}$, $x \geq 0$ check.

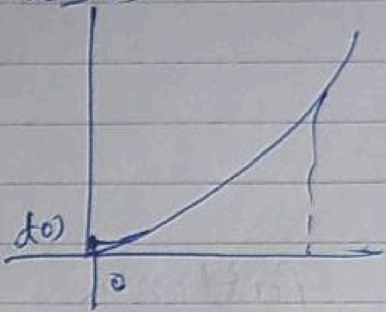
$h(x) = 1 + x \ln(x + \sqrt{1+x^2}) - \sqrt{1+x^2}$

$h'(x) = \ln(x + \sqrt{1+x^2}) + \cancel{\dfrac{x}{\sqrt{1+x^2}}} - \cancel{\dfrac{2x}{2\sqrt{1+x^2}}}$

$h'(x) = \ln(x + \sqrt{1+x^2}) \geq 0$ for $(x \geq 0)$

$h'(x) \geq 0$

$h(x)$ is $\uparrow$.




$f(x) > f(0)$

$1 + x \ln(x + \sqrt{1+x^2}) - \sqrt{1+x^2} > 0$

$1 + x \ln(x + \sqrt{1+x^2}) > \sqrt{1+x^2}$

Q. In $0 < x_1 < x_2 < \pi/2$ $\quad \dfrac{\tan x_2}{\tan x_1} > \dfrac{x_1}{x_2}$

$x_2 \tan x_2 > x_1 \tan x_1$

let $h(x) = x \tan x$

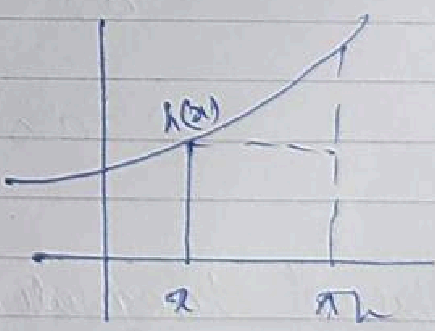
$h'(x) = \tan x + x \sec^2 x \geq 0$

$h'(x) \geq 0$

$h(x)$ is increasing.

So, if $x_2 > x_1$

$\Rightarrow h(x_2) > h(x_1)$




$x_2 \tan x_2 > x_1 \tan x_1$

$\dfrac{\tan x_2}{\tan x_1} > \dfrac{x_1}{x_2}$

Q. If $ax^2 + \frac{b}{x} \geq c$ $\forall x \in R$ then P.T

$$27ab^2 \geq 4c^3$$

$$h(x) = ax^2 + \frac{b}{x} - c$$

$$h'(x) = 2ax - \frac{b}{x^2} = 0$$

$$2ax = \frac{b}{x^2}$$

$$x^3 = \frac{b}{2a}$$

$$x = \left(\frac{b}{2a}\right)^{1/3}$$

Putting $x = \left(\frac{b}{2a}\right)^{1/3}$ in inequality

$$a\left(\frac{b}{2a}\right)^{2/3} + b\left(\frac{2a}{b}\right)^{1/3} \geq c$$

$$\cancel{a} \times \frac{b}{2\cancel{a}} + b \geq c\left(\frac{b}{2a}\right)^{1/3}$$

$$\frac{3b}{2} \geq c\left(\frac{b}{2a}\right)^{1/3}$$

$$\frac{27\,\cancel{b^3}\,b^2}{\cancel{8}\,4} \geq c\,\frac{\cancel{b}}{2a}$$

$$27ab^2 \geq c^2$$

** <u>Use of Inequality Method in deciding fxn's monotonic behaviour.</u>

① Normally $\frac{u}{v}$ type diff$^n$ is useful

(2) Take $N^r = h(x)$ & Use method of inequality.

Q. $f(x) = \dfrac{x}{\sin x}$ ; $x \in (0, 1)$ check it is ↑ or ↓ ?

(1) $f'(x) = \dfrac{\sin x - x\cos x}{\sin^2 x}$

(2) $h(x) = \sin x - x\cos x$

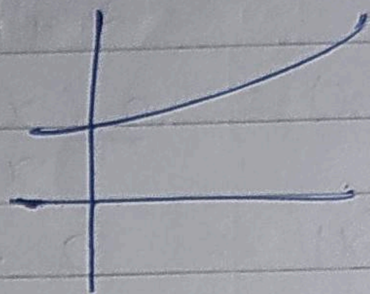
$h'(x) = \cancel{\cos x} + x \sin x - \cancel{\cos x}$

$h'(x) \geq 0$

$h(x)$ is ↑.

$h(x) > h(0)$

$\sin x - x\cos x > 0$



So, $f'(x) > 0$

$\Rightarrow f(x)$ is increasing

Q. Set of values a for $f(x) = x^3 + (a+2)x^2 + 3ax + 5$ is invertible?

$f'(x) = 3x^2 + 2(a+2)x + 3a > 0$ OR $< 0$

$\forall\, B > 0$

$D < 0$

$\cancel{4}(a+2)^2 - \cancel{4} \times 3a \times 3 > 0$

$a^2 + 4a + 4 - 9a > 0$

$a^2 - 5a + 4 > 0$

$(a-4)\ (a-1) > 0$

$1 < a < 4$

Q. Find 'a' if $f(x) = (a+2)x^3 - 3ax^2 + 9ax - 1$ ↓ for all $x$?

$f'(x) < 0$

$3(a+2)x^2 - 6ax + 9a < 0$

① 2) $3(a+2) < 0$

$a < -2$ — ①

② $D \leq 0$

$36a^2 - 108a(a+2) \leq 0$

$a^2 - 3a^2 - 6a \leq 0$

$-2a^2 - 6a \leq 0$

$a^2 + 3a \geq 0$

$a(a+3) \geq 0$

$a \in [-\infty, -3] \cup [0, \infty)$ — ②

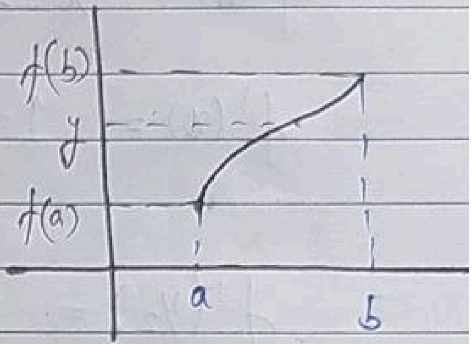
from ① & ②

$a \in [-\infty, -3]$

# Intermediate value Theorem (I·V·T)

Suppose $f : [a, b] \to R$ be a continous function.
Let, $y$ lie b/w $f(a)$ & $f(b)$.

Then, $\exists$ at least one ~~$x \in [a, b]$~~ $x \in (a, b)$
s.t $f(x) = y$.

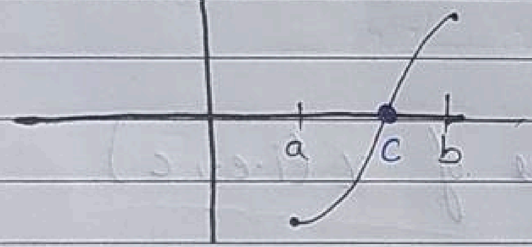


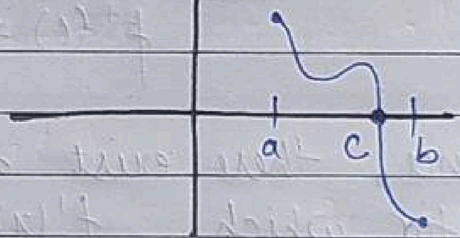

Q) Cond^n to apply IVT on any function.

Ans- The function must be continous in interval $[a, b]$

<u>Imp Result</u> : If $f(a) > 0$ & $f(b) < 0$ <u>OR</u> $f(a) < 0$ & $f(b) > 0$

then $\exists\ c \in (a, b)$ such that, $f(c) = 0$.




Q:) Prove that $e^{-x} = x$ has a solution b/w interval $(0, 1)$

Let, $g(x) = e^{-x} - x$.

$g(0) = e^{0} - 0 = 1$.

$g(1) = e^{-1} - 1 = \frac{1}{e} - 1 < 0$.

By IVT, $\exists\ c \in (0, 1)$ for which $g(x) = 0$.

$\Rightarrow \exists\ c \in (0, 1)$ such that $e^{c} = c$.

Q.) Show that there exist a critical point in interval $(1/2, 3/2)$ for function, $f(x) = x - \ln x$

$f(x) = x - \ln x$

$f'(x) = 1 - \frac{1}{x}$

$f'(1/2) = 1 - \frac{1}{1/2} = -1$

$f'(3/2) = 1 - \frac{1}{3/2} = \frac{1}{3}$

$f'(x)$ is continous function &

$f'(1/2) < 0, \quad f'(3/2) > 0$

so, $\exists \ c \in (1/2, 3/2)$

such that

$f'(c) = 0$.

Hence, there exist a value of $x$ (i.e., $c$) for which $f'(x) = 0$

$\Rightarrow$ There exist a critical point.

# Application of IVT:

**Theorem:** Every polynomial of odd degree must have at least one zero.

**Proof:**

Let $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$

where $n$ is odd.

$$f(x) = x^n \left( \frac{a_0}{x^n} + \frac{a_1}{x^{n-1}} + \frac{a_2}{x^{n-2}} + \cdots + a_n \right)$$

$\lim_{x \to +\infty} f(x) = \infty \quad \Big| \quad \lim_{x \to -\infty} f(x) = -\infty$

Hence, By IVT, $\exists c \in (-\infty, \infty)$

for which $f(c) = 0$.