(iv) A constant factor may be written either before or after the integral sign, i.e., $\int a\, f(x)\, dx = a \int f(x)\, dx$, where '$a$' is a constant.

(v) Properties (iii) and (iv) can be generalised to a finite number of functions $f_1, f_2, ..., f_n$ and the real numbers, $k_1, k_2, ..., k_n$ giving

$$\int (k_1 f_1(x) + k_2\, f_2(x) + ... + k_n f_n(x))\, dx = k_1 \int f_1(x) dx + k_2 \int f_2(x) dx + ... + k_n \int f_n(x) dx$$

7.1.5 *Methods of integration*

There are some methods or techniques for finding the integral where we can not directly select the antiderivative of function f by reducing them into standard forms. Some of these methods are based on

1. Integration by substitution
2. Integration using partial fractions
3. Integration by parts.

7.1.6 *Definite integral*

The definite integral is denoted by $\int_a^b f(x)\, dx$, where a is the lower limit of the integral and b is the upper limit of the integral. The definite integral is evaluated in the following two ways:

(i) The definite integral as the limit of the sum

(ii) $\int_a^b f(x)\, dx = \text{F}(b) - \text{F}(a)$, if F is an antiderivative of $f(x)$.

7.1.7 *The definite integral as the limit of the sum*

The definite integral $\int_a^b f(x)\, dx$ is the area bounded by the curve $y = f(x)$, the ordinates $x = a$, $x = b$ and the x-axis and given by

$$\int_a^b f(x)\, dx = (b - a) \lim_{n \to \infty} \frac{1}{n} \left[f(a) + f(a+h) + ... f(a + (n-1)h) \right]$$

or

$$\int_a^b f(x)\,dx = \lim_{h\to 0} h\left[f(a) + f(a+h) + \ldots + f\left(a+(n-1)h\right)\right],$$

where $h = \dfrac{b-a}{n} \to 0$ as $n \to \infty$.

7.1.8 *Fundamental Theorem of Calculus*

(i) *Area function* : The function A (x) denotes the area function and is given by A (x) = $\int_a^x f(x)\,dx$.

(ii) *First Fundamental Theorem of integral Calculus*

Let f be a continuous function on the closed interval [a, b] and let A (x) be the area function . Then A′ (x) = f (x) for all $x \in$ [a, b] .

(iii) *Second Fundamental Theorem of Integral Calculus*

Let f be continuous function defined on the closed interval [a, b] and F be an antiderivative of f.

$$\int_a^b f(x)\,dx = \left[F(x)\right]_a^b = F(b) - F(a).$$

7.1.9 *Some properties of Definite Integrals*

P_0 : $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$

P_1 : $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$, in particular, $\int_a^a f(x)\,dx = 0$

P_2 : $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

$$P_3 : \int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$$

$$P_4 : \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$$

$$P_5 : \int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$$

$$P_6 : \int_0^{2a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx, \text{if } f(2a-x) = f(x), \\ 0, \text{ if } f(2a-x) = -f(x). \end{cases}$$

P_7 : (i) $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$, if f is an even function i.e., $f(-x) = f(x)$

(ii) $\int_{-a}^{a} f(x)\,dx = 0$, if f is an odd function i.e., $f(-x) = -f(x)$

7.2 Solved Examples

Short Answer (S.A.)

Example 1 Integrate $\left(\frac{2a}{\sqrt{x}} - \frac{b}{x^2} + 3c\sqrt[3]{x^2}\right)$ w.r.t. x

Solution $\int\left(\frac{2a}{\sqrt{x}} - \frac{b}{x^2} + 3c\sqrt[3]{x^2}\right)dx$

$$= \int 2a(x)^{\frac{-1}{2}}\,dx - \int bx^{-2}\,dx + \int 3c\,x^{\frac{2}{3}}\,dx$$

$$= 4a\sqrt{x} + \frac{b}{x} + \frac{9cx^{\frac{5}{3}}}{5} + C\ .$$

Example 2 Evaluate $\int \frac{3ax}{b^2 + c^2x^2} dx$

Solution Let $v = b^2 + c^2x^2$, then $dv = 2c^2\, x dx$

Therefore, $\int \frac{3ax}{b^2 + c^2x^2} dx = \frac{3a}{2c^2} \int \frac{dv}{v}$

$$= \frac{3a}{2c^2} \log \left| b^2 + c^2x^2 \right| + C.$$

Example 3 Verify the following using the concept of integration as an antiderivative.

$$\int \frac{x^3 dx}{x+1} = x - \frac{x^2}{2} + \frac{x^3}{3} - \log|x+1| + C$$

Solution $\frac{d}{dx}\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \log|x+1| + C \right)$

$$= 1 - \frac{2x}{2} + \frac{3x^2}{3} - \frac{1}{x+1}$$

$$= 1 - x + x^2 - \frac{1}{x+1} = \frac{x^3}{x+1}.$$

Thus $\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \log|x+1| + C \right) = \int \frac{x^3}{x+1} dx$

Example 4 Evaluate $\int \sqrt{\frac{1+x}{1-x}}\, dx,\ x \neq 1.$

Solution Let $I = \int \sqrt{\frac{1+x}{1-x}}\, dx = \int \frac{1}{\sqrt{1-x^2}} dx + \int \frac{x\,dx}{\sqrt{1-x^2}} = \sin^{-1} x + I_1,$

where $I_1 = \dfrac{x\,dx}{\sqrt{1-x^2}}$.

Put $1 - x^2 = t^2 \Rightarrow -2x\,dx = 2t\,dt$. Therefore

$$I_1 = - \quad dt = -t + C = -\sqrt{1-x^2} + C$$

Hence $\quad I = \sin^{-1}x - \sqrt{1-x^2} + C$.

Example 5 Evaluate $\displaystyle\int \frac{dx}{\sqrt{(x-\alpha)(\beta-x)}}$, $\beta > \alpha$

Solution Put $x - \alpha = t^2$. Then $\beta - x = \beta - (t^2 + \alpha) = \beta - t^2 - \alpha = -t^2 - \alpha + \beta$ and $dx = 2t\,dt$. Now

$$I = \int \frac{2t\,dt}{\sqrt{t^2(\beta - \alpha - t^2)}} = \int \frac{2\,dt}{\sqrt{(\beta - \alpha - t^2)}}$$

$$= 2 \quad \frac{dt}{\sqrt{k^2 - t^2}}, \text{ where } k^2 = \beta - \alpha$$

$$= 2\sin^{-1}\frac{t}{k} + C = 2\sin^{-1}\sqrt{\frac{x-\alpha}{\beta-\alpha}} + C.$$

Example 6 Evaluate $\displaystyle\int \tan^8 x \sec^4 x\,dx$

Solution $I = \displaystyle\int \tan^8 x \sec^4 x\,dx$

$$= \int \tan^8 x\left(\sec^2 x\right)\sec^2 x\,dx$$

$$= \int \tan^8 x\left(\tan^2 x + 1\right)\sec^2 x\,dx$$

$$= \int \tan^{10} x \sec^2 x\,dx + \int \tan^8 x \sec^2 x\,dx$$

$$= \frac{\tan^{11} x}{11} + \frac{\tan^9 x}{9} + C.$$

Example 7 Find $\int \frac{x^3}{x^4 + 3x^2 + 2} dx$

Solution Put $x^2 = t$. Then $2x\,dx = dt$.

Now $$I = \int \frac{x^3 dx}{x^4 + 3x^2 + 2} = \frac{1}{2} \int \frac{t\,dt}{t^2 + 3t + 2}$$

Consider $$\frac{t}{t^2 + 3t + 2} = \frac{A}{t+1} + \frac{B}{t+2}$$

Comparing coefficient, we get A = –1, B = 2.

Then $$I = \frac{1}{2}\left[2\int \frac{dt}{t+2} - \int \frac{dt}{t+1}\right]$$

$$= \frac{1}{2}\left[2\log|t+2| - \log|t+1|\right]$$

$$= \log\left|\frac{x^2+2}{\sqrt{x^2+1}}\right| + C$$

Example 8 Find $\int \frac{dx}{2\sin^2 x + 5\cos^2 x}$

Solution Dividing numerator and denominator by $\cos^2 x$, we have

$$I = \int \frac{\sec^2 x\,dx}{2\tan^2 x + 5}$$

Put $\tan x = t$ so that $\sec^2 x\, dx = dt$. Then

$$\text{I} = \int \frac{dt}{2t^2 + 5} = \frac{1}{2}\int \frac{dt}{t^2 + \left(\sqrt{\frac{5}{2}}\right)^2}$$

$$= \frac{1}{2}\frac{\sqrt{2}}{\sqrt{5}}\tan^{-1}\left(\frac{\sqrt{2}t}{\sqrt{5}}\right) + \text{C}$$

$$= \frac{1}{\sqrt{10}}\tan^{-1}\left(\frac{\sqrt{2}\tan x}{\sqrt{5}}\right) + \text{C}.$$

Example 9 Evaluate $\int_{-1}^{2}(7x-5)\,dx$ as a limit of sums.

Solution Here $a = -1$, $b = 2$, and $h = \frac{2+1}{n}$, i.e, $nh = 3$ and $f(x) = 7x - 5$.

Now, we have

$$\int_{-1}^{2}(7x-5)\,dx = \lim_{h\to 0} h\left[f(-1) + f(-1+h) + f(-1+2h) + ... + f(-1+(n-1)h)\right]$$

Note that

$f(-1) = -7 - 5 = -12$

$f(-1 + h) = -7 + 7h - 5 = -12 + 7h$

$f(-1 + (n-1)h) = 7(n-1)h - 12.$

Therefore,

$$\int_{-1}^{2}(7x-5)\,dx = \lim_{h\to 0} h\left[(-12) + (7h-12) + (14h-12) + ... + (7(n-1)h-12)\right].$$

$$= \lim_{h\to 0} h\left[7h\left[1 + 2 + ... + (n-1)\right] - 12n\right]$$

$$= \lim_{h \to 0} h \left[7h \frac{(n-1)n}{2} - .12n \right] = \lim_{h \to 0} \left[\frac{7}{2}(nh)(nh-h) - 12nh \right]$$

$$= \frac{7}{2}(3)(3-0) - 12 \times 3 = \frac{7 \times 9}{2} - 36 = \frac{-9}{2}.$$

Example 10 Evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\tan^7 x}{\cot^7 x + \tan^7 x} dx$

Solution We have

$$\text{I} = \int_0^{\frac{\pi}{2}} \frac{\tan^7 x}{\cot^7 x + \tan^7 x} dx \qquad \text{...(1)}$$

$$= \int_0^{\frac{\pi}{2}} \frac{\tan^7\left(\frac{\pi}{2} - x\right)}{\cot^7\left(\frac{\pi}{2} - x\right) + \tan^7\left(\frac{\pi}{2} - x\right)} dx \qquad \text{by } (\text{P}_4)$$

$$= \int_0^{\frac{\pi}{2}} \frac{\cot^7(x)\, dx}{\cot^7 x\, dx + \tan^7 x} \qquad \text{...(2)}$$

Adding (1) and (2), we get

$$2\text{I} = \int_0^{\frac{\pi}{2}} \left(\frac{\tan^7 x + \cot^7 x}{\tan^7 x + \cot^7 x} \right) dx$$

$$= \int_0^{\frac{\pi}{2}} dx \text{ which gives } \text{I} = \frac{\pi}{4}.$$

Example 11 Find $\int_2^8 \frac{\sqrt{10-x}}{\sqrt{x}+\sqrt{10-x}}dx$

Solution We have

$$I = \int_2^8 \frac{\sqrt{10-x}}{\sqrt{x}+\sqrt{10-x}}dx \qquad ...(1)$$

$$= \int_2^8 \frac{\sqrt{10-(10-x)}}{\sqrt{10-x}+\sqrt{10-(10-x)}}dx \qquad \text{by } (P_3)$$

$$\Rightarrow \quad I = \int_2^8 \frac{\sqrt{x}}{\sqrt{10-x}+\sqrt{x}}dx \qquad (2)$$

Adding (1) and (2), we get

$$2I = \int_2^8 1dx = 8-2=6$$

Hence $\quad I = 3$

Example 12 Find $\int_0^{\frac{\pi}{4}} \sqrt{1+\sin 2x}\,dx$

Solution We have

$$I = \int_0^{\frac{\pi}{4}} \sqrt{1+\sin 2x}\,dx = \int_0^{\frac{\pi}{4}} \sqrt{(\sin x+\cos x)^2}\,dx$$

$$= \int_0^{\frac{\pi}{4}} (\sin x+\cos x)\,dx$$

$$= \left(-\cos x + \sin x\right)_0^{\frac{\pi}{4}}$$

$$I = 1.$$

Example 13 Find $\int x^2 \tan^{-1} x \, dx$.

Solution $I = \int x^2 \tan^{-1} x \, dx$

$$= \tan^{-1} x \int x^2 dx - \int \frac{1}{1+x^2} \cdot \frac{x^3}{3} dx$$

$$= \frac{x^3}{3} \tan^{-1} x - \frac{1}{3} \int \left(x - \frac{x}{1+x^2} \right) dx$$

$$= \frac{x^3}{3} \tan^{-1} x - \frac{x^2}{6} + \frac{1}{6} \log\left|1+x^2\right| + C.$$

Example 14 Find $\int \sqrt{10 - 4x + 4x^2} \, dx$

Solution We have

$$I = \int \sqrt{10 - 4x + 4x^2} \, dx \quad = \int \sqrt{(2x-1)^2 + (3)^2} \, dx$$

Put $t = 2x - 1$, then $dt = 2dx$.

Therefore, $I = \frac{1}{2} \int \sqrt{t^2 + (3)^2} \, dt$

$$= \frac{1}{2} t \frac{\sqrt{t^2+9}}{2} + \frac{9}{4} \log\left|t + \sqrt{t^2+9}\right| + C$$

$$= \frac{1}{4}(2x-1)\sqrt{(2x-1)^2+9} + \frac{9}{4} \log\left|(2x-1) + \sqrt{(2x-1)^2+9}\right| + C.$$

Long Answer (L.A.)

Example 15 Evaluate $\int \frac{x^2\,dx}{x^4+x^2-2}$.

Solution Let $x^2 = t$. Then

$$\frac{x^2}{x^4+x^2-2} = \frac{t}{t^2+t-2} = \frac{t}{(t+2)\,(t-1)} = \frac{\text{A}}{t+2} + \frac{\text{B}}{t-1}$$

So $\quad t = \text{A}\,(t-1) + \text{B}\,(t+2)$

Comparing coefficients, we get $\text{A} = \frac{2}{3},\ \text{B} = \frac{1}{3}$.

So $$\frac{x^2}{x^4+x^2-2} = \frac{2}{3}\frac{1}{x^2+2} + \frac{1}{3}\frac{1}{x^2-1}$$

Therefore,

$$\int \frac{x^2}{x^4+x^2-2}\,dx = \frac{2}{3}\int \frac{1}{x^2+2}\,dx + \frac{1}{3}\int \frac{dx}{x^2-1}$$

$$= \frac{2}{3}\frac{1}{\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}} + \frac{1}{6}\log\left|\frac{x-1}{x+1}\right| + \text{C}$$

Example16 Evaluate $\int \frac{x^3+x}{x^4 \quad 9}\,dx$

Solution We have

$$\text{I} = \int \frac{x^3+x}{x^4 \quad 9}\,dx \quad = \int \frac{x^3}{x^4 \quad 9}\,dx + \frac{x\,dx}{x^4 \quad 9} \quad = \text{I}_1 + \text{I}_2\,.$$

Now $\quad I_1 = \int \frac{x^3}{x^4 - 9}$

Put $\quad t = x^4 - 9$ so that $4x^3\, dx = dt$. Therefore

$$I_1 = \frac{1}{4}\int \frac{dt}{t} = \frac{1}{4}\log|t| + C_1 = \frac{1}{4}\log\left|x^4 - 9\right| + C_1$$

Again, $\quad I_2 = \int \frac{x\,dx}{x^4 - 9}$.

Put $\quad x^2 = u$ so that $2x\, dx = du$. Then

$$I_2 = \frac{1}{2}\int \frac{du}{u^2 - (3)^2} = \frac{1}{2 \times 6}\log\left|\frac{u-3}{u+3}\right| + C_2$$

$$= \frac{1}{12}\log\left|\frac{x^2-3}{x^2+3}\right| + C_2.$$

Thus $\quad I = I_1 + I_2$

$$= \frac{1}{4}\log\left|x^4 - 9\right| + \frac{1}{12}\log\left|\frac{x^2-3}{x^2+3}\right| + C.$$

Example 17 Show that $\int_0^{\frac{\pi}{2}} \frac{\sin^2 x}{\sin x + \cos x} = \frac{1}{\sqrt{2}}\log(\sqrt{2}+1)$

Solution We have

$$I = \int_0^{\frac{\pi}{2}} \frac{\sin^2 x}{\sin x + \cos x}\,dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sin^2\left(\frac{\pi}{2}-x\right)}{\sin\left(\frac{\pi}{2}-x\right)+\cos\left(\frac{\pi}{2}-x\right)}\,dx \qquad \text{(by P4)}$$

$\Rightarrow$ $$\text{I} = \int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{\sin x+\cos x}\,dx$$

Thus, we get $$2\text{I} = \frac{1}{\sqrt{2}}\int_0^{\frac{\pi}{2}} \frac{dx}{\cos\left(x-\frac{\pi}{4}\right)}$$

$$= \frac{1}{\sqrt{2}}\int_0^{\frac{\pi}{2}} \sec\left(x-\frac{\pi}{4}\right)dx = \frac{1}{\sqrt{2}}\left[\log\left(\sec\left(x-\frac{\pi}{4}\right)+\tan\left(x-\frac{\pi}{4}\right)\right)\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{\sqrt{2}}\left[\log\left(\sec\frac{\pi}{4}+\tan\frac{\pi}{4}\right)-\log\sec\left(-\frac{\pi}{4}\right)+\tan\left(-\frac{\pi}{4}\right)\right]$$

$$= \frac{1}{\sqrt{2}}\left[\log\left(\sqrt{2}+1\right)-\log\left(\sqrt{2}-1\right)\right] \quad = \frac{1}{\sqrt{2}}\log\left|\frac{\sqrt{2}+1}{\sqrt{2}-1}\right|$$

$$= \frac{1}{\sqrt{2}}\log\left(\frac{(\sqrt{2}+1)^2}{1}\right) \quad = \frac{2}{\sqrt{2}}\log\left(\sqrt{2}+1\right)$$

Hence $$\text{I} = \frac{1}{\sqrt{2}}\log\left(\sqrt{2}+1\right).$$

Example 18 Find $\int_0^1 x\left(\tan^{-1}x\right)^2 dx$

Solution $I = \int_0^1 x\left(\tan^{-1} x\right)^2 dx$.

Integrating by parts, we have

$$I = \frac{x^2}{2}\left[\left(\tan^{-1} x\right)^2\right]_0^1 - \frac{1}{2}\int_0^1 x^2 . 2\frac{\tan^{-1} x}{1+x^2}\, dx$$

$$= \frac{\pi^2}{32} - \int_0^1 \frac{x^2}{1+x^2} . \tan^{-1} x\, dx$$

$$= \frac{\pi^2}{32} - I_1 \text{, where } I_1 = \int_0^1 \frac{x^2}{1+x^2} \tan^{-1} x dx$$

Now $$I_1 = \int_0^1 \frac{x^2+1-1}{1+x^2} \tan^{-1} x\, dx$$

$$= \int_0^1 \tan^{-1} x\, dx - \int_0^1 \frac{1}{1+x^2} \tan^{-1} x\, dx$$

$$= I_2 - \frac{1}{2}\left(\left(\tan^{-1} x\right)^2\right)_0^1 = I_2 - \frac{\pi^2}{32}$$

Here $$I_2 = \int_0^1 \tan^{-1} x\, dx = \left(x \tan^{-1} x\right)_0^1 - \int_0^1 \frac{x}{1+x^2}\, dx$$

$$= \frac{\pi}{4} - \frac{1}{2}\left(\log\left|1+x^2\right|\right)_0^1 = \frac{\pi}{4} - \frac{1}{2}\log 2 .$$

Thus $$I_1 = \frac{\pi}{4} - \frac{1}{2}\log 2 - \frac{\pi^2}{32}$$

Therefore, $$I = \frac{\pi^2}{32} - \frac{\pi}{4} + \frac{1}{2}\log 2 + \frac{\pi^2}{32} = \frac{\pi^2}{16} - \frac{\pi}{4} + \frac{1}{2}\log 2$$

$$= \frac{\pi^2 - 4\pi}{16} + \log\sqrt{2}\,.$$

Example 19 Evaluate $\int_{-1}^{2} f(x)\,dx$, where $f(x) = |x + 1| + |x| + |x - 1|$.

Solution We can redefine f as $f(x) = \begin{cases} 2 - x, & \text{if } -1 < x \le 0 \\ x + 2, & \text{if } 0 < x \le 1 \\ 3x, & \text{if } 1 < x \le 2 \end{cases}$

Therefore, $$\int_{-1}^{2} f(x)\,dx = \int_{-1}^{0} (2 - x)\,dx + \int_{0}^{1} (x + 2)\,dx + \int_{1}^{2} 3x\,dx \qquad \text{(by } P_2\text{)}$$

$$= \left(2x - \frac{x^2}{2}\right)_{-1}^{0} + \left(\frac{x^2}{2} + 2x\right)_{0}^{1} + \left(\frac{3x^2}{2}\right)_{1}^{2}$$

$$= 0 - \left(-2 - \frac{1}{2}\right) + \left(\frac{1}{2} + 2\right) + 3\left(\frac{4}{2} - \frac{1}{2}\right) = \frac{5}{2} + \frac{5}{2} + \frac{9}{2} = \frac{19}{2}\,.$$

Objective Type Questions

Choose the correct answer from the given four options in each of the Examples from 20 to 30.

Example 20 $\int e^x (\cos x - \sin x)\,dx$ is equal to

(A) $e^x \cos x + C$ (B) $e^x \sin x + C$

(C) $-e^x \cos x + C$ (D) $-e^x \sin x + C$

Solution (A) is the correct answer since $\int e^x \left[f(x) + f'(x) \right] dx = e^x f(x) + C$. Here $f(x) = \cos x$, $f'(x) = -\sin x$.

Example 21 $\int \frac{dx}{\sin^2 x \cos^2 x}$ is equal to

(A) $\tan x + \cot x + C$ . . . . $x + \cot x)^2 + C$

(C) $\tan x - \cot x + C$ (D) $(\tan x - \cot x)^2 + C$

Solution (C) is the correct answer, since

$$I = \int \frac{dx}{\sin^2 x \cos^2 x} = \int \frac{(\sin^2 x + \cos^2 x)\,dx}{\sin^2 x \cos^2 x}$$

$$= \int \sec^2 x\, dx + \int \operatorname{cosec}^2 x\, dx = \tan x - \cot x + C$$

Example 22 If $\int \frac{3e^x - 5e^{-x}}{4e^x + 5e^{-x}} dx = ax + b \log |4e^x + 5e^{-x}| + C$, then

(A) $a = \frac{-1}{8},\ b = \frac{7}{8}$ (B) $a = \frac{1}{8},\ b = \frac{7}{8}$

(C) $a = \frac{-1}{8},\ b = \frac{-7}{8}$ (D) $a = \frac{1}{8},\ b = \frac{-7}{8}$

Solution (C) is the correct answer, since differentiating both sides, we have

$$\frac{3e^x - 5e^{-x}}{4e^x + 5e^{-x}} = a + b\,\frac{(4e^x - 5e^{-x})}{4e^x + 5e^{-x}},$$

giving $3e^x - 5e^{-x} = a(4e^x + 5e^{-x}) + b(4e^x - 5e^{-x})$. Comparing coefficients on both sides, we get $3 = 4a + 4b$ and $-5 = 5a - 5b$. This verifies $a = \frac{-1}{8},\ b = \frac{7}{8}$.

Example 23 $\int_{a+c}^{b+c} f(x)\,dx$ is equal to

(A) $\int_a^b f(x-c)\,dx$ (B) $\int_a^b f(x+c)\,dx$

(C) $\int_a^b f(x)\,dx$ (D) $\int_{a-c}^{b-c} f(x)\,dx$

Solution (B) is the correct answer, since by putting $x = t + c$, we get

$$\text{I} = \int_a^b f(c+t)\,dt = \int_a^b f(x+c)\,dx.$$

Example 24 If f and g are continuous functions in [0, 1] satisfying $f(x) = f(a-x)$ and $g(x) + g(a-x) = a$, then $\int_0^a f(x).g(x)\,dx$ is equal to

(A) $\frac{a}{2}$ (B) $\frac{a}{2}\int_0^a f(x)\,dx$

(C) $\int_0^a f(x)\,dx$ (D) $a\int_0^a f(x)\,dx$

Solution B is the correct answer. Since $\text{I} = \int_0^a f(x).g(x)\,dx$

$$= \int_0^a f(a-x)\,g(a-x)\,dx = \int_0^a f(x)\,(a-g(x))\,dx$$

$$= a\int_0^a f(x)\,dx - \int_0^a f(x).g(x)\,dx = a\int_0^a f(x)\,dx - \text{I}$$

or $\quad \mathrm{I} = \frac{a}{2}\int_0^a f(x)\,dx$.

Example 25 If $x = \int_0^y \frac{dt}{\sqrt{1+9t^2}}$ and $\frac{d^2y}{dx^2} = ay$, then a is equal to

(A) 3 (B) 6 (C) 9 (D) 1

Solution (C) is the correct answer, since $x = \int_0^y \frac{dt}{\sqrt{1+9t^2}} \Rightarrow \frac{dx}{dy} = \frac{1}{\sqrt{1+9y^2}}$

which gives $\frac{d^2y}{dx^2} = \frac{18y}{2\sqrt{1+9y^2}} \cdot \frac{dy}{dx} = 9y.$

Example 26 $\int_{-1}^{1} \frac{x^3+|x|+1}{x^2+2|x|+1}\,dx$ is equal to

(A) log 2 (B) 2 log 2 (C) $\frac{1}{2}\log 2$ (D) 4 log 2

Solution (B) is the correct answer, since I = $\int_{-1}^{1} \frac{x^3+|x|+1}{x^2+2|x|+1}\,dx$

$$= \int_{-1}^{1} \frac{x^3}{x^2+2|x|+1} + \int_{-1}^{1} \frac{|x|+1}{x^2+2|x|+1}\,dx \quad = \quad 0 + 2\int_0^1 \frac{|x|+1}{(|x|+1)^2}\,dx$$

[odd function + even function]

$$= 2\int_0^1 \frac{x+1}{(x+1)^2}\,dx = 2\int_0^1 \frac{1}{x+1}\,dx \qquad = 2\left|\log|x+1|\right|_0^1 = 2\log 2.$$

Example 27 If $\int_0^1 \frac{e^t}{1+t}dt = a$, then $\int_0^1 \frac{e^t}{(1+t)^2}dt$ is equal to

(A) $a - 1 + \frac{e}{2}$ (B) $a + 1 - \frac{e}{2}$ (C) $a - 1 - \frac{e}{2}$ (D) $a + 1 + \frac{e}{2}$

Solution (B) is the correct answer, since I = $\int_0^1 \frac{e^t}{1+t}\, dt$

$$= \left|\frac{1}{1+t}e^t\right|_0^1 + \int_0^1 \frac{e^t}{(1+t)^2}dt = a \text{ (given)}$$

Therefore, $\int_0^1 \frac{e^t}{(1+t)^2} = a - \frac{e}{2} + 1.$

Example 28 $\int_{-2}^{2} |x\cos \pi x|\, dx$ is equal to

(A) $\frac{8}{\pi}$ (B) $\frac{4}{\pi}$ (C) $\frac{2}{\pi}$ (D) $\frac{1}{\pi}$

Solution (A) is the correct answer, since I = $\int_{-2}^{2} |x\cos \pi x|\, dx = 2\int_0^2 |x\cos \pi x|\, dx$

$$= 2\left\{\int_0^{\frac{1}{2}} |x\cos \pi x|\, dx + \int_{\frac{1}{2}}^{\frac{3}{2}} |x\cos \pi x|\, dx + \int_{\frac{3}{2}}^{2} |x\cos \pi x|\, dx\right\} = \frac{8}{\pi}.$$

Fill in the blanks in each of the Examples 29 to 32.

Example 29 $\int \frac{\sin^6 x}{\cos^8 x}\, dx =$ _______.

Solution $\dfrac{\tan^7 x}{7} + C$

Example 30 $\int_{-a}^{a} f(x)\, dx = 0$ if f is an ________ function.

Solution Odd.

Example 31 $\int_{0}^{2a} f(x)\, dx = 2\int_{0}^{a} f(x)\, dx$, if $f(2a - x) =$ ________.

Solution $f(x)$.

Example 32 $\int_{0}^{\frac{\pi}{2}} \dfrac{\sin^n x\, dx}{\sin^n x + \cos^n x} =$ ________.

Solution $\dfrac{\pi}{4}$.

7.3 EXERCISE

Short Answer (S.A.)

Verify the following :

1. $\int \dfrac{2x-1}{2x+3} dx = x - \log |(2x + 3)^2| + C$

2. $\int \dfrac{2x+3}{x^2 + 3x} dx = \log |x^2 + 3x| + C$

Evaluate the following:

3. $\int \dfrac{(x^2 + 2)\, dx}{x+1}$

4. $\int \dfrac{e^{6\log x} - e^{5\log x}}{e^{4\log x} - e^{3\log x}} dx$