4.5 Magnetic Field due to a Current Element, Biot-Savart Law

All magnetic fields that we know are due to currents (or moving charges) and due to intrinsic magnetic moments of particles. Here, we shall study the relation between current and the magnetic field it produces. It is given by the Biot-Savart's law. Figure 4.9 shows a finite conductor XY carrying current I. Consider an infinitesimal element dl of the conductor. The magnetic field d**B** due to this element is to be determined at a point P which is at a distance r from it. Let θ be the angle between d$\mathbf{l}$ and the displacement vector **r**. According to Biot-Savart's law, the magnitude of the magnetic field d**B** is proportional to the current I, the element length $|d\mathbf{l}|$, and inversely proportional to the square of the distance r. Its direction* is perpendicular to the plane containing d$\mathbf{l}$ and **r**. Thus, in vector notation,

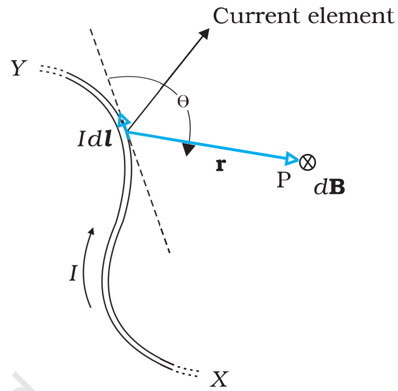

FIGURE 4.9 Illustration of the Biot-Savart law. The current element $I\,d\mathbf{l}$ produces a field d**B** at a distance r. The ⊗ sign indicates that the field is perpendicular to the plane of this page and directed into it.

$$d\mathbf{B} \propto \frac{I\,d\mathbf{l}\times\mathbf{r}}{r^3}$$

$$= \frac{\mu_0}{4\pi}\frac{I\,d\mathbf{l}\times\mathbf{r}}{r^3} \qquad [4.11(a)]$$

where $\mu_0/4\pi$ is a constant of proportionality. The above expression holds when the medium is vacuum.

The magnitude of this field is,

$$|d\mathbf{B}| = \frac{\mu_0}{4\pi}\frac{I\,dl\sin\theta}{r^2} \qquad [4.11(b)]$$

where we have used the property of cross-product. Equation [4.11 (a)] constitutes our basic equation for the magnetic field. The proportionality constant in SI units has the exact value,

$$\frac{\mu_0}{4\pi} = 10^{-7}\,\text{Tm/A} \qquad [4.11(c)]$$

We call μ_0 the *permeability* of free space (or vacuum).

The Biot-Savart law for the magnetic field has certain similarities, as well as, differences with the Coulomb's law for the electrostatic field. Some of these are:

(i) Both are long range, since both depend inversely on the square of distance from the source to the point of interest. The principle of superposition applies to both fields. [In this connection, note that the magnetic field is *linear* in the *source* $I\,d\mathbf{l}$ just as the electrostatic field is linear in its source: the electric charge.]

(ii) The electrostatic field is produced by a scalar source, namely, the electric charge. The magnetic field is produced by a vector source $I\,d\mathbf{l}$.

* The sense of $d\mathbf{l}\times\mathbf{r}$ is also given by the *Right Hand Screw rule* : Look at the plane containing vectors d$\mathbf{l}$ and **r**. Imagine moving from the first vector towards second vector. If the movement is anticlockwise, the resultant is towards you. If it is clockwise, the resultant is away from you.

(iii) The electrostatic field is along the displacement vector joining the source and the field point. The magnetic field is perpendicular to the plane containing the displacement vector **r** and the current element $I\,\mathrm{d}\boldsymbol{l}$.

(iv) There is an angle dependence in the Biot-Savart law which is not present in the electrostatic case. In Fig. 4.9, the magnetic field at any point in the direction of d$\boldsymbol{l}$ (the dashed line) is zero. Along this line, $\theta = 0$, $\sin\theta = 0$ and from Eq. [4.11(a)], $|\mathrm{d}\mathbf{B}| = 0$.

There is an interesting relation between ε_0, the permittivity of free space; μ_0, the permeability of free space; and c, the speed of light in vacuum:

$$\varepsilon_0\mu_0 = (4\pi\varepsilon_0)\frac{\mu_0}{4\pi} = \frac{1}{9\times10^9}\left(10^{-7}\right) = \frac{1}{(3\times10^8)^2} = \frac{1}{c^2}$$

We will discuss this connection further in Chapter 8 on the electromagnetic waves. Since the speed of light in vacuum is constant, the product $\mu_0\varepsilon_0$ is fixed in magnitude. Choosing the value of either ε_0 or μ_0, fixes the value of the other. In SI units, μ_0 is fixed to be equal to $4\pi \times 10^{-7}$ in magnitude.

Example 4.5 An element $\Delta\boldsymbol{l} = \Delta x\,\hat{\mathbf{i}}$ is placed at the origin and carries a large current I = 10 A (Fig. 4.10). What is the magnetic field on the y-axis at a distance of 0.5 m. Δx = 1 cm.

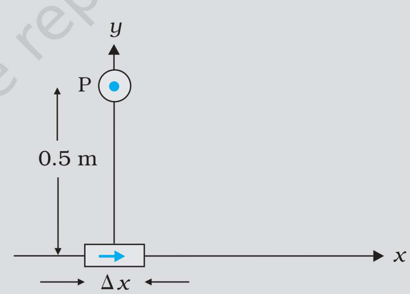

FIGURE 4.10

Solution

$$|\mathrm{d}\mathbf{B}| = \frac{\mu_0}{4\pi}\frac{I\,\mathrm{d}l\sin\theta}{r^2} \quad \text{[using Eq. (4.11)]}$$

$$\mathrm{d}l = \Delta x = 10^{-2}\,\mathrm{m}, \; I = 10\,\mathrm{A}, \; r = 0.5\,\mathrm{m} = y, \; \mu_0/4\pi = 10^{-7}\,\frac{\mathrm{T\,m}}{\mathrm{A}}$$

$\theta = 90°$; $\sin\theta = 1$

$$|\mathrm{d}\mathbf{B}| = \frac{10^{-7}\times10\times10^{-2}}{25\times10^{-2}} = 4\times10^{-8}\,\mathrm{T}$$

The direction of the field is in the +z-direction. This is so since,

$$\mathrm{d}\boldsymbol{l}\times\mathbf{r} = \Delta x\,\hat{\mathbf{i}}\times y\,\hat{\mathbf{j}} = y\,\Delta x\left(\hat{\mathbf{i}}\times\hat{\mathbf{j}}\right) = y\,\Delta x\,\hat{\mathbf{k}}$$

We remind you of the following cyclic property of cross-products,

$$\hat{\mathbf{i}}\times\hat{\mathbf{j}} = \hat{\mathbf{k}}; \; \hat{\mathbf{j}}\times\hat{\mathbf{k}} = \hat{\mathbf{i}}; \; \hat{\mathbf{k}}\times\hat{\mathbf{i}} = \hat{\mathbf{j}}$$

Note that the field is small in magnitude.

EXAMPLE 4.5

In the next section, we shall use the Biot-Savart law to calculate the magnetic field due to a circular loop.

4.6 Magnetic Field on the Axis of a Circular Current Loop

In this section, we shall evaluate the magnetic field due to a circular coil along its axis. The evaluation entails summing up the effect of infinitesimal current elements ($I\,d\boldsymbol{l}$) mentioned in the previous section. We assume that the current I is steady and that the evaluation is carried out in free space (i.e., vacuum).

Figure 4.11 depicts a circular loop carrying a steady current I. The loop is placed in the y-z plane with its centre at the origin O and has a radius R. The x-axis is the axis of the loop. We wish to calculate the magnetic field at the point P on this axis. Let x be the distance of P from the centre O of the loop.

Consider a conducting element $d\boldsymbol{l}$ of the loop. This is shown in Fig. 4.11. The magnitude dB of the magnetic field due to $d\boldsymbol{l}$ is given by the Biot-Savart law [Eq. 4.11(a)],

$$dB = \frac{\mu_0}{4\pi}\frac{I\left|d\boldsymbol{l}\times\mathbf{r}\right|}{r^3} \qquad (4.12)$$

Now $r^2 = x^2 + R^2$. Further, any element of the loop will be perpendicular to the displacement vector from the element to the axial point. For example, the element $d\boldsymbol{l}$ in Fig. 4.11 is in the y-z plane, whereas, the displacement vector $\mathbf{r}$ from $d\boldsymbol{l}$ to the axial point P is in the x-y plane. Hence $|d\boldsymbol{l}\times\mathbf{r}| = r\,dl$. Thus,

$$dB = \frac{\mu_0}{4\pi}\frac{I\,dl}{\left(x^2+R^2\right)} \qquad (4.13)$$

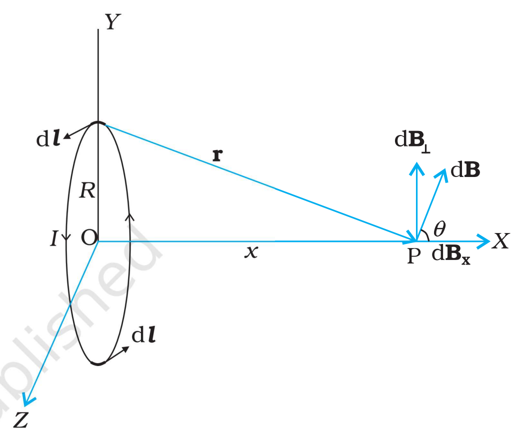

FIGURE 4.11 Magnetic field on the axis of a current carrying circular loop of radius R. Shown are the magnetic field $d\mathbf{B}$ (due to a line element $d\boldsymbol{l}$) and its components along and perpendicular to the axis.

The direction of $d\mathbf{B}$ is shown in Fig. 4.11. It is perpendicular to the plane formed by $d\boldsymbol{l}$ and $\mathbf{r}$. It has an x-component $d\mathbf{B}_x$ and a component perpendicular to x-axis, $d\mathbf{B}_\perp$. When the components perpendicular to the x-axis are summed over, they cancel out and we obtain a null result. For example, the $d\mathbf{B}_\perp$ component due to $d\boldsymbol{l}$ is cancelled by the contribution due to the diametrically opposite $d\boldsymbol{l}$ element, shown in Fig. 4.11. Thus, only the x-component survives. The net contribution along x-direction can be obtained by integrating $dB_x = dB\cos\theta$ over the loop. For Fig. 4.11,

$$\cos\theta = \frac{R}{(x^2+R^2)^{1/2}} \qquad (4.14)$$

From Eqs. (4.13) and (4.14),

$$dB_x = \frac{\mu_0 I\,dl}{4\pi}\frac{R}{\left(x^2+R^2\right)^{3/2}}$$

The summation of elements dl over the loop yields $2\pi R$, the circumference of the loop. Thus, the magnetic field at P due to entire circular loop is

$$\mathbf{B} = B_x\hat{\mathbf{i}} = \frac{\mu_0 I R^2}{2\left(x^2 + R^2\right)^{3/2}}\hat{\mathbf{i}} \tag{4.15}$$

As a special case of the above result, we may obtain the field at the centre of the loop. Here $x = 0$, and we obtain,

$$\mathbf{B}_0 = \frac{\mu_0 I}{2R}\hat{\mathbf{i}} \tag{4.16}$$

The magnetic field lines due to a circular wire form closed loops and are shown in Fig. 4.12. The direction of the magnetic field is given by (another) *right-hand thumb rule* stated below:

Curl the palm of your right hand around the circular wire with the fingers pointing in the direction of the current. The right-hand thumb gives the direction of the magnetic field.

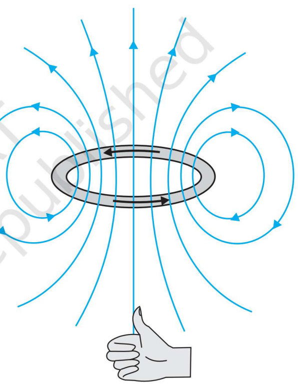

FIGURE 4.12 The magnetic field lines for a current loop. The direction of the field is given by the right-hand thumb rule described in the text. The upper side of the loop may be thought of as the north pole and the lower side as the south pole of a magnet.

EXAMPLE 4.6

Example 4.6 A straight wire carrying a current of 12 A is bent into a semi-circular arc of radius 2.0 cm as shown in Fig. 4.13(a). Consider the magnetic field **B** at the centre of the arc. (a) What is the magnetic field due to the straight segments? (b) In what way the contribution to **B** from the semicircle differs from that of a circular loop and in what way does it resemble? (c) Would your answer be different if the wire were bent into a semi-circular arc of the same radius but in the opposite way as shown in Fig. 4.13(b)?

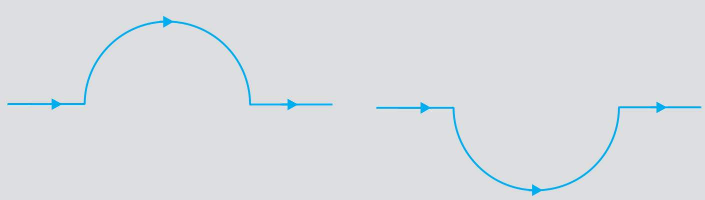

FIGURE 4.13

Solution

(a) d**l** and **r** for each element of the straight segments are parallel. Therefore, d***l*** × **r** = 0. Straight segments do not contribute to |**B**|.

(b) For all segments of the semicircular arc, d***l*** × **r** are all parallel to each other (into the plane of the paper). All such contributions add up in magnitude. Hence direction of **B** for a semicircular arc is given by the right-hand rule and magnitude is half that of a circular loop. Thus **B** is 1.9 × 10^{-4} T normal to the plane of the paper going into it.

(c) Same magnitude of **B** but opposite in direction to that in (b).

EXAMPLE 4.6

Example 4.7 Consider a tightly wound 100 turn coil of radius 10 cm, carrying a current of 1 A. What is the magnitude of the magnetic field at the centre of the coil?

Solution Since the coil is tightly wound, we may take each circular element to have the same radius R = 10 cm = 0.1 m. The number of turns N = 100. The magnitude of the magnetic field is,

$$B = \frac{\mu_0 NI}{2R} = \frac{4\pi \times 10^{-7} \times 10^2 \times 1}{2 \times 10^{-1}} = 2\pi \times 10^{-4} = 6.28 \times 10^{-4} \text{ T}$$

EXAMPLE 4.7

4.7 AMPERE'S CIRCUITAL LAW

There is an alternative and appealing way in which the Biot-Savart law may be expressed. Ampere's circuital law considers an open surface with a boundary (Fig. 4.14). The surface has current passing through it. We consider the boundary to be made up of a number of small line elements. Consider one such element of length *dl*. We take the value of the tangential component of the magnetic field, B_t, at this element and multiply it by the length of that element *dl* [Note: $B_t dl = \mathbf{B} \cdot d\boldsymbol{l}$]. All such products are added together. We consider the limit as the lengths of elements get smaller and their number gets larger. The sum then tends to an integral. Ampere's law states that this integral is equal to μ_0 times the total current passing through the surface, i.e.,

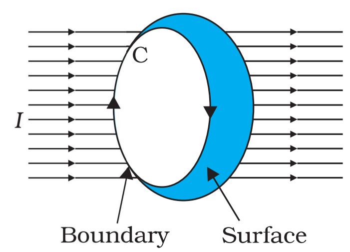

FIGURE 4.14

$$\oint \mathbf{B} \cdot d\boldsymbol{l} = \mu_0 I \qquad [4.17(a)]$$

where I is the total current through the surface. The integral is taken over the closed loop coinciding with the boundary C of the surface. The relation above involves a sign-convention, given by the right-hand rule. Let the fingers of the right-hand be curled in the sense the boundary is traversed in the loop integral $\oint \mathbf{B} \cdot d\boldsymbol{l}$. Then the direction of the thumb gives the sense in which the current I is regarded as positive.

For several applications, a much simplified version of Eq. [4.17(a)] proves sufficient. We shall assume that, in such cases, it is possible to choose the loop (called an *amperian loop*) such that at each point of the loop, *either*

(i) **B** is tangential to the loop and is a non-zero *constant* B, *or*

(ii) **B** is normal to the loop, *or*

(iii) **B** vanishes.

Now, let L be the length (part) of the loop for which **B** is tangential. Let I_e be the current enclosed by the loop. Then, Eq. (4.17) reduces to,

$$BL = \mu_0 I_e \qquad [4.17(b)]$$

When there is a system with a symmetry such as for a *straight infinite current-carrying wire* in Fig. 4.15, the Ampere's law enables an easy evaluation of the magnetic field, much the same way Gauss' law helps in determination of the electric field. This is exhibited in the Example 4.9 below. The boundary of the loop chosen is a circle and magnetic field is tangential to the circumference of the circle. The law gives, for the left hand side of Eq. [4.17 (b)], B. $2\pi r$. We find that the magnetic field at a distance r outside the wire is *tangential* and given by

$$B \times 2\pi r = \mu_0 I,$$

$$B = \mu_0 I / (2\pi r) \qquad (4.18)$$

The above result for the infinite wire is interesting from several points of view.

(i) It implies that the field at every point on a circle of radius r, (with the wire along the axis), is same in magnitude. In other words, the magnetic field possesses what is called a *cylindrical symmetry*. The field that normally can depend on three coordinates depends only on one: r. Whenever there is symmetry, the solutions simplify.

(ii) The field direction at any point on this circle is tangential to it. Thus, the lines of constant magnitude of magnetic field form concentric circles. Notice now, in Fig. 4.1(c), the iron filings form concentric circles. These lines called *magnetic field lines* form closed loops. This is unlike the electrostatic field lines which originate from positive charges and end at negative charges. The expression for the magnetic field of a straight wire provides a theoretical justification to Oersted's experiments.

(iii) Another interesting point to note is that even though the wire is infinite, the field due to it at a non-zero distance is *not* infinite. It tends to blow up only when we come very close to the wire. The field is directly proportional to the current and inversely proportional to the distance from the (infinitely long) current source.

ANDRE AMPERE (1775 –1836)

Andre Ampere (1775 – 1836) Andre Marie Ampere was a French physicist, mathematician and chemist who founded the science of electrodynamics. Ampere was a child prodigy who mastered advanced mathematics by the age of 12. Ampere grasped the significance of Oersted's discovery. He carried out a large series of experiments to explore the relationship between current electricity and magnetism. These investigations culminated in 1827 with the publication of the 'Mathematical Theory of Electrodynamic Phenomena Deduced Solely from Experiments'. He hypothesised that *all* magnetic phenomena are due to circulating electric currents. Ampere was humble and absent-minded. He once forgot an invitation to dine with the Emperor Napoleon. He died of pneumonia at the age of 61. His gravestone bears the epitaph: *Tandem Felix* (Happy at last).

(iv) There exists a simple rule to determine the direction of the magnetic field due to a long wire. This rule, called the *right-hand rule**, is:

Grasp the wire in your right hand with your extended thumb pointing in the direction of the current. Your fingers will curl around in the direction of the magnetic field.

Ampere's circuital law is not new in content from Biot-Savart law. Both relate the magnetic field and the current, and both express the same physical consequences of a steady electrical current. Ampere's law is to Biot-Savart law, what Gauss's law is to Coulomb's law. Both, Ampere's and Gauss's law relate a physical quantity on the periphery or boundary (magnetic or electric field) to another physical quantity, namely, the source, in the interior (current or charge). We also note that Ampere's circuital law holds for steady currents which do not fluctuate with time. The following example will help us understand what is meant by the term *enclosed* current.

Example 4.8 Figure 4.15 shows a long straight wire of a circular cross-section (radius a) carrying steady current I. The current I is uniformly distributed across this cross-section. Calculate the magnetic field in the region $r < a$ and $r > a$.

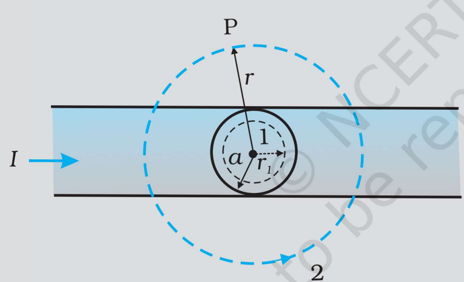

FIGURE 4.15

Solution (a) Consider the case $r > a$. The Amperian loop, labelled 2, is a circle concentric with the cross-section. For this loop,

$L = 2\pi r$

I_e = Current enclosed by the loop = I

The result is the familiar expression for a long straight wire

$B(2\pi r) = \mu_0 I$

$$B = \frac{\mu_0 I}{2\pi r} \qquad [4.19(a)]$$

$$B \propto \frac{1}{r} \qquad (r > a)$$

(b) Consider the case $r < a$. The Amperian loop is a circle labelled 1. For this loop, taking the radius of the circle to be r,

$L = 2\pi r$

EXAMPLE 4.8

* Note that there are *two distinct* right-hand rules: One which gives the direction of **B** on the axis of current-loop and the other which gives direction of **B** for a straight conducting wire. Fingers and thumb play different roles in the two.

EXAMPLE 4.8

Now the current enclosed I_e is not I, but is less than this value. Since the current distribution is uniform, the current enclosed is,

$$I_e = I\left(\frac{\pi r^2}{\pi a^2}\right) = \frac{Ir^2}{a^2}$$

Using Ampere's law, $B(2\pi r) = \mu_0 \dfrac{I r^2}{a^2}$

$$B = \left(\frac{\mu_0 I}{2\pi a^2}\right) r \qquad [4.19(b)]$$

$B \propto r \qquad (r < a)$

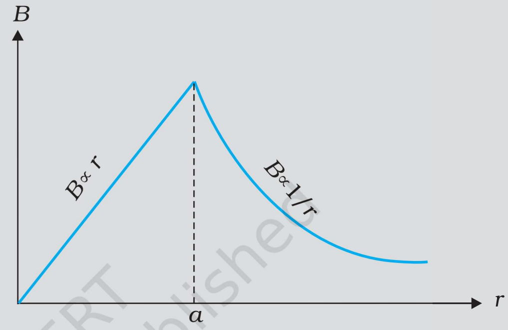

FIGURE 4.16

Figure (4.16) shows a plot of the magnitude of **B** with distance r from the centre of the wire. The direction of the field is tangential to the respective circular loop (1 or 2) and given by the right-hand rule described earlier in this section.

This example possesses the required symmetry so that Ampere's law can be applied readily.

It should be noted that while Ampere's circuital law holds for any loop, it may not always facilitate an evaluation of the magnetic field in every case. For example, for the case of the circular loop discussed in Section 4.6, it cannot be applied to extract the simple expression $B = \mu_0 I/2R$ [Eq. (4.16)] for the field at the centre of the loop. However, there exists a large number of situations of high symmetry where the law can be conveniently applied. We shall use it in the next section to calculate the magnetic field produced by two commonly used and very useful magnetic systems: the *solenoid* and the *toroid*.

4.8 The Solenoid and the Toroid

The solenoid and the toroid are two pieces of equipment which generate magnetic fields. The synchrotron uses a combination of both to generate the high magnetic fields required. In both, solenoid and toroid, we come across a situation of high symmetry where Ampere's law can be conveniently applied.