It is given that $\frac{1}{2^n \sin\alpha}$, 1, $2^n \sin\alpha$ are in ~~AP~~.

AP. for some value of α, $\alpha \in \left(0, \frac{\pi}{2}\right)$.

Let say for $n=1$, α satisfying the above A.P. is α_1, for $n=2$, the value ~~of~~ is α_2 and so on.

if $S = \sum_{r=1}^{\infty} \sin\alpha_r$, then the value of S is

a) 1 b) $\frac{1}{2}$ c) 2 d) None

<u>solution:-</u>

since $\frac{1}{2^n \sin\alpha}$, 1, $2^n \sin\alpha$ are in A.P.

$$\therefore\ 2 = \frac{1}{2^n \sin\alpha} + 2^n \sin\alpha \quad \left(\because \text{ if } a, b, c \text{ are in AP} \Rightarrow 2b = a+c\right)$$

since we have reciprocal system which points to AM≥GM inequality

now, $2^n > 0$ & $\because \alpha \in \left(0, \frac{\pi}{2}\right) \Rightarrow \sin\alpha > 0$

$\therefore\ 2^n \sin\alpha > 0$

now, consider AM of $2^n \sin\alpha$ and $\frac{1}{2^n \sin\alpha}$

$$AM = \frac{2^n \sin\alpha + \frac{1}{2^n \sin\alpha}}{2} = \frac{2}{2} = 1$$

$$GM = \sqrt{\left(2^n \sin\alpha\right)\left(\frac{1}{2^n \sin\alpha}\right)} = 1$$

since, AM = GM

$\Rightarrow$ terms are equal.

$$\therefore\ 2^n \sin\alpha = \frac{1}{2^n \sin\alpha}$$

$$\Rightarrow (\sin\alpha)^2 = \frac{1}{(2^n)^2} \Rightarrow \sin\alpha = \frac{1}{2^n}$$

for, $n=1$ $\quad \sin\alpha_1 = \frac{1}{2^1}$

$n=2$ $\quad \sin\alpha_2 = \frac{1}{2^2}$

$n=3$ $\quad \sin\alpha_3 = \frac{1}{2^3}$

......

& for $n=\infty$ $\quad \sin\alpha_\infty = \frac{1}{2^\infty}$

now, $S = \sum_{r=1}^{\infty} \sin\alpha_r = \sin\alpha_1 + \sin\alpha_2 + \dots + \sin\alpha_\infty$

$$= \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots + \frac{1}{2^\infty}$$

$$= \frac{1/2}{1-1/2} = 1 \quad \left(\because S_\infty = \frac{a}{1-r}\right)$$

$\therefore S = 1$

Main thing in this question was to understand ~~the~~ a that this is problem of AM, GM inequality. since we have reciprocal system and its sum was also given which points towards AM & GM inequality.

Reciprocal system means if you have terms like $bx, \frac{a}{x}$ $\quad where\ a, b \in R$

wherein if we multiply the two numbers we should get rid of the variable x.

NOTE 1 : here *a* and *b* are some real constants and *x* is a variable

NOTE 2 : if AM=GM or AM=HM or GM=HM then, the terms involved in the means will be equal to each other.

Similar questions can be asked in quadratic equations and expressions chapter
(using (AM GM) , (GM HM) , (AM HM) inequalities).
For example :

$\boldsymbol{If\ the\ equation\ x^4 - 4x^3 + cx^2 + dx + 1 = 0}$
$\boldsymbol{has\ three\ positive\ roots\ then\ find\ the\ value\ of}$:

$$\frac{1}{(root\_1)^2} + \frac{1}{(root\_2)^2} + \frac{1}{(root\_3)^2} + \frac{1}{(root\_4)^2}$$

SOLUTION :

$x^4 - 4x^3 + cx^2 + dx + 1 = 0$

let it's roots be $\alpha, \beta, \gamma, \delta$

now, $\alpha\beta\gamma\delta = 1$

since, we have given in the question that there are 3 positive roots,

let those 3 positive roots be α, β, γ

$\alpha\beta\gamma\delta = 1$

$\alpha, \beta, \gamma > 0 \Rightarrow \cancel{\alpha, \beta, \gamma >} \ \alpha\beta\gamma > 0$

$\Rightarrow \delta > 0$

because (+ve number) (+ve number) (+ve number) (+ve number) = (+ve number)

$\therefore$ all roots are positive

now, sum of roots, $\alpha+\beta+\gamma+\delta = 4$

product of roots, $\alpha\beta\gamma\delta = 1$

since, $\alpha, \beta, \gamma, \delta > 0$

$\therefore \; AM = \dfrac{\alpha+\beta+\gamma+\delta}{4} = \dfrac{4}{4} = 1$

$\cancel{GM = (\alpha\beta\gamma\delta)^{4}}$ $GM = (\alpha\beta\gamma\delta)^{1/4} = 1^{1/4} = 1$

(roots)

now, $AM = GM \Rightarrow$ all terms are equal

$\Rightarrow \alpha = \beta = \gamma = \delta$

$\alpha+\beta+\gamma+\delta = 4$

$\Rightarrow 4\alpha = 4$

$\Rightarrow \alpha = 1$

$\Rightarrow \alpha = \beta = \gamma = \delta = 1$

$\therefore \dfrac{1}{\alpha^2} + \dfrac{1}{\beta^2} + \dfrac{1}{\gamma^2} + \dfrac{1}{\delta^2} = 1+1+1+1 = 4$