1. Let $\Delta = \begin{vmatrix} 1+\cos^2\theta & \sin^2\theta & 4\cos 6\theta \\ \cos^2\theta & 1+\sin^2\theta & 4\cos 6\theta \\ \cos^2\theta & \sin^2\theta & 1+4\cos 6\theta \end{vmatrix} = 0$

Applying $C_1 \to C_1 + C_2$, we get

$$\Delta = \begin{vmatrix} 2 & \sin^2\theta & 4\cos 6\theta \\ 2 & 1+\sin^2\theta & 4\cos 6\theta \\ 1 & \sin^2\theta & 1+4\cos 6\theta \end{vmatrix} = 0$$

Applying $R_1 \to R_1 - 2R_3$ and $R_2 \to R_2 - 2R_3$, we get

$$\Delta = \begin{vmatrix} 0 & -\sin^2\theta & -2-4\cos 6\theta \\ 0 & 1-\sin^2\theta & -2-4\cos 6\theta \\ 1 & \sin^2\theta & 1+4\cos 6\theta \end{vmatrix} = 0$$

On expanding w.r.t. C_1, we get

$\Rightarrow \sin^2\theta\,(2+4\cos 6\theta) + (2+4\cos 6\theta)(1-\sin^2\theta) = 0$

$\Rightarrow 2+4\cos 6\theta = 0 \Rightarrow \cos 6\theta = -\frac{1}{2} = \cos\frac{2\pi}{3}$

$\Rightarrow \quad 6\theta = \frac{2\pi}{3} \Rightarrow \theta = \frac{\pi}{9} \quad \left[\because \theta \in \left(0, \frac{\pi}{3}\right)\right]$

2. Given equation

$$\begin{vmatrix} x & -6 & -1 \\ 2 & -3x & x-3 \\ -3 & 2x & x+2 \end{vmatrix} = 0$$

On expansion of determinant along R_1, we get

$x\,[(-3x)(x+2) - 2x(x-3)] + 6\,[2(x+2) + 3(x-3)]$
$\quad -1\,[2(2x) - (-3x)(-3)] = 0$

$\Rightarrow x\,[-3x^2 - 6x - 2x^2 + 6x] + 6[2x + 4 + 3x - 9]$
$\quad -1\,[4x - 9x] = 0$

$\Rightarrow x(-5x^2) + 6(5x-5) - 1(-5x) = 0$

$\Rightarrow \quad -5x^3 + 30x - 30 + 5x = 0$

$\Rightarrow \quad 5x^3 - 35x + 30 = 0 \quad \Rightarrow \quad x^3 - 7x + 6 = 0.$

Since all roots are real

$\therefore$ Sum of roots $= -\dfrac{\text{coefficient of } x^2}{\text{coefficient of } x^3} = 0$

3. Given determinants are

$$\Delta_1 = \begin{vmatrix} x & \sin\theta & \cos\theta \\ -\sin\theta & -x & 1 \\ \cos\theta & 1 & x \end{vmatrix}$$

$= -x^3 + \sin\theta\cos\theta - \sin\theta\cos\theta + x\cos^2\theta - x + x\sin^2\theta$

$= -x^3$

and $\Delta_2 = \begin{vmatrix} x & \sin 2\theta & \cos 2\theta \\ -\sin 2\theta & -x & 1 \\ \cos 2\theta & 1 & x \end{vmatrix}, x \neq 0$

$= -x^3$ (similarly as Δ_1)

So, according to options, we get $\Delta_1 + \Delta_2 = -2x^3$

4. Given

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}\cdots\begin{bmatrix} 1 & n-1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 78 \\ 0 & 1 \end{bmatrix}$$

$\because \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2+1 \\ 0 & 1 \end{bmatrix}$,

$\begin{bmatrix} 1 & 2+1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3+2+1 \\ 0 & 1 \end{bmatrix}$,

$\vdots \qquad \vdots \qquad \vdots$

$\therefore \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}\cdots\begin{bmatrix} 1 & n-1 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & (n-1)+(n-2)+\ldots+3+2+1 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & \dfrac{n(n-1)}{2} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 78 \\ 0 & 1 \end{bmatrix}$

Since, both matrices are equal, so equating corresponding element, we get

$\dfrac{n(n-1)}{2} = 78 \Rightarrow n(n-1) = 156$

$= 13 \times 12 = 13(13-1)$

$\Rightarrow \quad n = 13$

So, $\quad A = \begin{bmatrix} 1 & 13 \\ 0 & 1 \end{bmatrix} = A^{-1} = \begin{bmatrix} 1 & -13 \\ 0 & 1 \end{bmatrix}$

$\left[\because \text{if } |A| = 1 \text{ and } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \text{then } A^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\right]$

5. Given, quadratic equation is $x^2 + x + 1 = 0$ having roots α, β.

Then, $\alpha + \beta = -1$ and $\alpha\beta = 1$

Now, given determinant

$$\Delta = \begin{vmatrix} y+1 & \alpha & \beta \\ \alpha & y+\beta & 1 \\ \beta & 1 & y+\alpha \end{vmatrix}$$

On applying $R_1 \to R_1 + R_2 + R_3$, we get

$$\Delta = \begin{vmatrix} y+1+\alpha+\beta & y+1+\alpha+\beta & y+1+\alpha+\beta \\ \alpha & y+\beta & 1 \\ \beta & 1 & y+\alpha \end{vmatrix}$$

$= \begin{vmatrix} y & y & y \\ \alpha & y+\beta & 1 \\ \beta & 1 & y+\alpha \end{vmatrix} \qquad [\because \alpha+\beta = -1]$

On applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$, we get

$$\Delta = \begin{vmatrix} y & 0 & 0 \\ \alpha & y+\beta-\alpha & 1-\alpha \\ \beta & 1-\beta & y+\alpha-\beta \end{vmatrix}$$

$= y[(y+(\beta-\alpha))(y-(\beta-\alpha)) - (1-\alpha)(1-\beta)]$

[expanding along R_1]

$= y[y^2 - (\beta-\alpha)^2 - (1-\alpha-\beta+\alpha\beta)]$

$= y[y^2 - \beta^2 - \alpha^2 + 2\alpha\beta - 1 + (\alpha+\beta) - \alpha\beta]$

$= y[y^2 - (\alpha+\beta)^2 + 2\alpha\beta + 2\alpha\beta - 1 + (\alpha+\beta) - \alpha\beta]$

$= y[y^2 - 1 + 3 - 1 - 1] = y^3 \quad [\because \alpha+\beta = -1 \text{ and } \alpha\beta = 1]$

6. Given, matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & b & c \\ 4 & b^2 & c^2 \end{bmatrix}$, so

$$\det(A) = \begin{vmatrix} 1 & 1 & 1 \\ 2 & b & c \\ 4 & b^2 & c^2 \end{vmatrix}$$

On applying, $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$,

we get $\det(A) = \begin{vmatrix} 1 & 0 & 0 \\ 2 & b-2 & c-2 \\ 4 & b^2-4 & c^2-4 \end{vmatrix}$

$= \begin{vmatrix} b-2 & c-2 \\ b^2-4 & c^2-4 \end{vmatrix}$

$= \begin{vmatrix} b-2 & c-2 \\ (b-2)(b+2) & (c-2)(c+2) \end{vmatrix}$

$= (b-2)(c-2)\begin{vmatrix} 1 & 1 \\ b+2 & c+2 \end{vmatrix}$

[taking common $(b-2)$ from C_1 and $(c-2)$ from C_2]

$= (b-2)(c-2)(c-b)$

Since, 2, b and c are in AP, if assume common difference of AP is d, then

$b = 2 + d$ and $c = 2 + 2d$

So, $\quad |A| = d(2d)d = 2d^3 \in [2, 16] \qquad$ [given]

$\Rightarrow \quad d^3 \in [1, 8] \Rightarrow d \in [1, 2]$

$\therefore \quad 2 + 2d \in [2+2, 2+4]$

$= [4, 6] \Rightarrow c \in [4, 6]$

7. Given matrix $A = \begin{bmatrix} 1 & \sin\theta & 1 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{bmatrix}$

$\Rightarrow \det(A) = |A| = \begin{vmatrix} 1 & \sin\theta & 1 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{vmatrix}$

$= 1(1+\sin^2\theta) - \sin\theta(-\sin\theta + \sin\theta) + 1(\sin^2\theta + 1)$

$\Rightarrow |A| = 2(1+\sin^2\theta) \qquad \ldots\text{(i)}$

As we know that, for $\theta \in \left(\dfrac{3\pi}{4}, \dfrac{5\pi}{4}\right)$

$\sin\theta \in \left(-\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}\right)$

$= -x^3 + \sin\theta\cos\theta - \sin\theta\cos\theta + x\cos^2\theta - x + x\sin^2\theta$

$= -x^3$

and $\Delta_2 = \begin{vmatrix} x & \sin 2\theta & \cos 2\theta \\ -\sin 2\theta & -x & 1 \\ \cos 2\theta & 1 & x \end{vmatrix}, x \neq 0$

$= -x^3$ (similarly as Δ_1)

So, according to options, we get $\Delta_1 + \Delta_2 = -2x^3$

4. Given

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}\cdots\begin{bmatrix} 1 & n-1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 78 \\ 0 & 1 \end{bmatrix}$$

$\because \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2+1 \\ 0 & 1 \end{bmatrix},$

$\begin{bmatrix} 1 & 2+1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3+2+1 \\ 0 & 1 \end{bmatrix},$

$\vdots \qquad \vdots \qquad \vdots$

$\therefore \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}\cdots\begin{bmatrix} 1 & n-1 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & (n-1)+(n-2)+\ldots+3+2+1 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & \dfrac{n(n-1)}{2} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 78 \\ 0 & 1 \end{bmatrix}$

Since, both matrices are equal, so equating corresponding element, we get

$\dfrac{n(n-1)}{2} = 78 \Rightarrow n(n-1) = 156$

$= 13 \times 12 = 13(13-1)$

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So, $\quad A = \begin{bmatrix} 1 & 13 \\ 0 & 1 \end{bmatrix} = A^{-1} = \begin{bmatrix} 1 & -13 \\ 0 & 1 \end{bmatrix}$

$\left[\because \text{if } |A| = 1 \text{ and } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \text{ then } A^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\right]$

5. Given, quadratic equation is $x^2 + x + 1 = 0$ having roots α, β.

Then, $\alpha + \beta = -1$ and $\alpha\beta = 1$

Now, given determinant

$$\Delta = \begin{vmatrix} y+1 & \alpha & \beta \\ \alpha & y+\beta & 1 \\ \beta & 1 & y+\alpha \end{vmatrix}$$

On applying $R_1 \to R_1 + R_2 + R_3$, we get

$$\Delta = \begin{vmatrix} y+1+\alpha+\beta & y+1+\alpha+\beta & y+1+\alpha+\beta \\ \alpha & y+\beta & 1 \\ \beta & 1 & y+\alpha \end{vmatrix}$$

$= \begin{vmatrix} y & y & y \\ \alpha & y+\beta & 1 \\ \beta & 1 & y+\alpha \end{vmatrix} \qquad [\because \alpha + \beta = -1]$

On applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$, we get

$$\Delta = \begin{vmatrix} y & 0 & 0 \\ \alpha & y+\beta-\alpha & 1-\alpha \\ \beta & 1-\beta & y+\alpha-\beta \end{vmatrix}$$

$= y[(y+(\beta-\alpha))(y-(\beta-\alpha)) - (1-\alpha)(1-\beta)]$

[expanding along R_1]

$= y[y^2 - (\beta-\alpha)^2 - (1-\alpha-\beta+\alpha\beta)]$

$= y[y^2 - \beta^2 - \alpha^2 + 2\alpha\beta - 1 + (\alpha+\beta) - \alpha\beta]$

$= y[y^2 - (\alpha+\beta)^2 + 2\alpha\beta + 2\alpha\beta - 1 + (\alpha+\beta) - \alpha\beta]$

$= y[y^2 - 1 + 3 - 1 - 1] = y^3 \quad [\because \alpha + \beta = -1 \text{ and } \alpha\beta = 1]$

6. Given, matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & b & c \\ 4 & b^2 & c^2 \end{bmatrix}$, so

$$\det(A) = \begin{vmatrix} 1 & 1 & 1 \\ 2 & b & c \\ 4 & b^2 & c^2 \end{vmatrix}$$

On applying, $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$,

we get $\det(A) = \begin{vmatrix} 1 & 0 & 0 \\ 2 & b-2 & c-2 \\ 4 & b^2-4 & c^2-4 \end{vmatrix}$

$= \begin{vmatrix} b-2 & c-2 \\ b^2-4 & c^2-4 \end{vmatrix}$

$= \begin{vmatrix} b-2 & c-2 \\ (b-2)(b+2) & (c-2)(c+2) \end{vmatrix}$

$= (b-2)(c-2)\begin{vmatrix} 1 & 1 \\ b+2 & c+2 \end{vmatrix}$

[taking common $(b-2)$ from C_1 and $(c-2)$ from C_2]

$= (b-2)(c-2)(c-b)$

Since, 2, b and c are in AP, if assume common difference of AP is d, then

$b = 2 + d$ and $c = 2 + 2d$

So, $\quad |A| = d(2d)d = 2d^3 \in [2, 16] \qquad$ [given]

$\Rightarrow \quad d^3 \in [1, 8] \quad \Rightarrow \quad d \in [1, 2]$

$\therefore \quad 2 + 2d \in [2+2, 2+4]$

$= [4, 6] \quad \Rightarrow \quad c \in [4, 6]$

7. Given matrix $A = \begin{bmatrix} 1 & \sin\theta & 1 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{bmatrix}$

$\Rightarrow \det(A) = |A| = \begin{vmatrix} 1 & \sin\theta & 1 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{vmatrix}$

$= 1(1 + \sin^2\theta) - \sin\theta(-\sin\theta + \sin\theta) + 1(\sin^2\theta + 1)$

$\Rightarrow |A| = 2(1 + \sin^2\theta) \qquad \ldots(i)$

As we know that, for $\theta \in \left(\dfrac{3\pi}{4}, \dfrac{5\pi}{4}\right)$

$\sin\theta \in \left(-\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}\right)$

$$\therefore\ |A| = \begin{vmatrix} -2 & 4+d & (\sin\theta)-2 \\ 1 & (\sin\theta)+2 & d \\ 5 & (2\sin\theta)-d & (-\sin\theta)+2+2d \end{vmatrix}$$

$$= \begin{vmatrix} -2 & 4+d & (\sin\theta)-2 \\ 1 & (\sin\theta)+2 & d \\ 1 & 0 & 0 \end{vmatrix}$$

$(R_3 \to R_3 - 2R_2 + R_1)$

$= 1\,[(4+d)d - (\sin\theta+2)(\sin\theta-2)]$

(expanding along R_3)

$= (d^2 + 4d - \sin^2\theta + 4)$

$= (d^2 + 4d + 4) - \sin^2\theta$

$= (d+2)^2 - \sin^2\theta$

Note that $|A|$ will be minimum if $\sin^2\theta$ is maximum i.e. if $\sin^2\theta$ takes value 1.

$\because\quad |A|_{\min} = 8,$

therefore $\quad (d+2)^2 - 1 = 8$

$\Rightarrow \quad (d+2)^2 = 9$

$\Rightarrow \quad d+2 = \pm 3$

$\Rightarrow \quad d = 1, -5$

12. Given,

$$\begin{vmatrix} x-4 & 2x & 2x \\ 2x & x-4 & 2x \\ 2x & 2x & x-4 \end{vmatrix} = (A+Bx)(x-A)^2$$

$\Rightarrow$ Apply $C_1 \to C_1 + C_2 + C_3$

$$\begin{vmatrix} 5x-4 & 2x & 2x \\ 5x-4 & x-4 & 2x \\ 5x-4 & 2x & x-4 \end{vmatrix} = (A+Bx)(x-A)^2$$

Taking common $(5x-4)$ from C_1, we get

$$(5x-4)\begin{vmatrix} 1 & 2x & 2x \\ 1 & x-4 & 2x \\ 1 & 2x & x-4 \end{vmatrix} = (A+Bx)(x-A)^2$$

Apply $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$

$$\therefore (5x-4)\begin{vmatrix} 1 & 2x & 0 \\ 0 & -x-4 & 0 \\ 0 & 0 & -x-4 \end{vmatrix} = (A+Bx)(x-A)^2$$

Expanding along C_1, we get

$(5x-4)(x+4)^2 = (A+Bx)(x-A)^2$

Equating, we get, $A = -4$ and $B = 5$

13. Given, $\quad 2\omega + 1 = z$

$\Rightarrow \quad 2\omega + 1 = \sqrt{-3} \quad [\because z = \sqrt{-3}]$

$\Rightarrow \quad \omega = \dfrac{-1+\sqrt{3}i}{2}$

Since, ω is cube root of unity.

$\therefore \quad \omega^2 = \dfrac{-1-\sqrt{3}i}{2}$ and $\omega^{3n} = 1$

$$\text{Now, } \begin{vmatrix} 1 & 1 & 1 \\ 1 & -\omega^2-1 & \omega^2 \\ 1 & \omega^2 & \omega^7 \end{vmatrix} = 3k$$

$$\Rightarrow \begin{vmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{vmatrix} = 3k$$

$[\because 1 + \omega + \omega^2 = 0 \text{ and } \omega^7 = (\omega^3)^2 \cdot \omega = \omega]$

On applying $R_1 \to R_1 + R_2 + R_3$, we get

$$\begin{vmatrix} 3 & 1+\omega+\omega^2 & 1+\omega+\omega^2 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{vmatrix} = 3k$$

$$\Rightarrow \begin{vmatrix} 3 & 0 & 0 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{vmatrix} = 3k$$

$\Rightarrow \quad 3(\omega^2 - \omega^4) = 3k$

$\Rightarrow \quad (\omega^2 - \omega) = k$

$\therefore \quad k = \left(\dfrac{-1-\sqrt{3}i}{2}\right) - \left(\dfrac{-1+\sqrt{3}i}{2}\right) = -\sqrt{3}i = -z$

14. PLAN Use the property that, two determinants can be multiplied column-to-row or row-to-column, to write the given determinant as the product of two determinants and then expand.

Given, $f(n) = \alpha^n + \beta^n$, $f(1) = \alpha + \beta$, $f(2) = \alpha^2 + \beta^2$, $f(3) = \alpha^3 + \beta^3$, $f(4) = \alpha^4 + \beta^4$

$$\text{Let } \Delta = \begin{vmatrix} 3 & 1+f(1) & 1+f(2) \\ 1+f(1) & 1+f(2) & 1+f(3) \\ 1+f(2) & 1+f(3) & 1+f(4) \end{vmatrix}$$

$$\Rightarrow \Delta = \begin{vmatrix} 3 & 1+\alpha+\beta & 1+\alpha^2+\beta^2 \\ 1+\alpha+\beta & 1+\alpha^2+\beta^2 & 1+\alpha^3+\beta^3 \\ 1+\alpha^2+\beta^2 & 1+\alpha^3+\beta^3 & 1+\alpha^4+\beta^4 \end{vmatrix}$$

$$= \begin{vmatrix} 1\cdot1+1\cdot1+1\cdot1 & 1\cdot1+1\cdot\alpha+1\cdot\beta & 1\cdot1+1\cdot\alpha^2+1\cdot\beta^2 \\ 1\cdot1+1\cdot\alpha+1\cdot\beta & 1\cdot1+\alpha\cdot\alpha+\beta\cdot\beta & 1\cdot1+\alpha\cdot\alpha^2+\beta\cdot\beta^2 \\ 1\cdot1+1\cdot\alpha^2+1\cdot\beta^2 & 1\cdot1+\alpha^2\cdot\alpha+\beta^2\cdot\beta & 1\cdot1+\alpha^2\cdot\alpha^2+\beta^2\cdot\beta^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \beta \\ 1 & \alpha^2 & \beta^2 \end{vmatrix}\begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \beta \\ 1 & \alpha^2 & \beta^2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \beta \\ 1 & \alpha^2 & \beta^2 \end{vmatrix}^2$$

On expanding, we get $\Delta = (1-\alpha)^2(1-\beta)^2(\alpha-\beta)^2$

But given, $\quad \Delta = K(1-\alpha)^2(1-\beta)^2(\alpha-\beta)^2$

Hence, $K(1-\alpha)^2(1-\beta)^2(\alpha-\beta)^2 = (1-\alpha)^2(1-\beta)^2(\alpha-\beta)^2$

$\therefore \quad K = 1$

15. PLAN It is a simple question on scalar multiplication, i.e.

$$\begin{vmatrix} ka_1 & ka_2 & ka_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = k\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Description of Situation Construction of matrix,

i.e. if $a = [a_{ij}]_{3\times3} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Here, $P = [a_{ij}]_{3\times 3} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$Q = [b_{ij}]_{3\times 3} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$

where, $b_{ij} = 2^{i+j} a_{ij}$

$\therefore \quad |Q| = \begin{vmatrix} 4a_{11} & 8a_{12} & 16a_{13} \\ 8a_{21} & 16a_{22} & 32a_{23} \\ 16a_{31} & 32a_{32} & 64a_{33} \end{vmatrix}$

$= 4\times 8\times 16 \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ 2a_{21} & 2a_{22} & 2a_{23} \\ 4a_{31} & 4a_{32} & 4a_{33} \end{vmatrix}$

$= 2^9 \times 2\times 4 \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

$= 2^{12} \cdot |P| = 2^{12} \cdot 2 = 2^{13}$

16. We know, $\quad |A^n| = |A|^n$

Since, $\quad |A^3| = 125 \Rightarrow |A|^3 = 125$

$\Rightarrow \quad \begin{vmatrix} \alpha & 2 \\ 2 & \alpha \end{vmatrix} = 5 \quad \Rightarrow \alpha^2 - 4 = 5 \Rightarrow \alpha = \pm 3$

17. Given, $\begin{vmatrix} \sin x & \cos x & \cos x \\ \cos x & \sin x & \cos x \\ \cos x & \cos x & \sin x \end{vmatrix} = 0$

Applying $C_1 \to C_1 + C_2 + C_3$

$= \begin{vmatrix} \sin x + 2\cos x & \cos x & \cos x \\ \sin x + 2\cos x & \sin x & \cos x \\ \sin x + 2\cos x & \cos x & \sin x \end{vmatrix}$

$= (2\cos x + \sin x) \begin{vmatrix} 1 & \cos x & \cos x \\ 1 & \sin x & \cos x \\ 1 & \cos x & \sin x \end{vmatrix} = 0$

Applying $R_2 \to R_2 - R_1, R_3 \to R_3 - R_1$

$\Rightarrow (2\cos x + \sin x) \begin{vmatrix} 1 & \cos x & \cos x \\ 0 & \sin x - \cos x & 0 \\ 0 & 0 & \sin x - \cos x \end{vmatrix} = 0$

$\Rightarrow \quad (2\cos x + \sin x)(\sin x - \cos x)^2 = 0$

$\Rightarrow \quad 2\cos x + \sin x = 0$ or $\sin x - \cos x = 0$

$\Rightarrow \quad 2\cos x = -\sin x$ or $\sin x = \cos x$

$\Rightarrow \cot x = -1/2$ gives no solution in $-\frac{\pi}{4} \le x \le \frac{\pi}{4}$

and $\sin x = \cos x \Rightarrow \tan x = 1 \Rightarrow x = \pi/4$

18. Given,

$f(x) = \begin{vmatrix} 1 & x & x+1 \\ 2x & x(x-1) & (x+1)x \\ 3x(x-1) & x(x-1)(x-2) & (x+1)x(x-1) \end{vmatrix}$

Applying $C_3 \to C_3 - (C_1 + C_2)$

$= \begin{vmatrix} 1 & x & 0 \\ 2x & x(x-1) & 0 \\ 3x(x-1) & x(x-1)(x-2) & 0 \end{vmatrix} = 0$

$\therefore \quad f(x) = 0 \Rightarrow f(100) = 0$

19. Let $\Delta = \begin{vmatrix} 1 & a & a^2 \\ \cos(p-d)x & \cos px & \cos(p+d)x \\ \sin(p-d)x & \sin px & \sin(p+d)x \end{vmatrix}$

Applying $C_1 \to C_1 + C_3$

$\Rightarrow \Delta = \begin{vmatrix} 1+a^2 & a & a^2 \\ \cos(p-d)x + \cos(p+d)x & \cos px & \cos(p+d)x \\ \sin(p-d)x + \sin(p+d)x & \sin px & \sin(p+d)x \end{vmatrix}$

$\Rightarrow \quad \Delta = \begin{vmatrix} 1+a^2 & a & a^2 \\ 2\cos px \cos dx & \cos px & \cos(p+d)x \\ 2\sin px \cos dx & \sin px & \sin(p+d)x \end{vmatrix}$

Applying $C_1 \to C_1 - 2\cos dx\, C_2$

$\Rightarrow \quad \Delta = \begin{vmatrix} 1+a^2-2a\cos dx & a & a^2 \\ 0 & \cos px & \cos(p+d)x \\ 0 & \sin px & \sin(p+d)x \end{vmatrix}$

$\Rightarrow \quad \Delta = (1+a^2-2a\cos dx)\,[\sin(p+d)x\cos px - \sin px\cos(p+d)x]$

$\Rightarrow \quad \Delta = (1+a^2-2a\cos dx)\sin dx$

which is independent of p.

20. Given, $\begin{vmatrix} xp+y & x & y \\ yp+z & y & z \\ 0 & xp+y & yp+z \end{vmatrix} = 0$

Applying $\; C_1 \to C_1 - (pC_2 + C_3)$

$\Rightarrow \quad \begin{vmatrix} 0 & x & y \\ 0 & y & z \\ -(xp^2+yp+yp+z) & xp+y & yp+z \end{vmatrix} = 0$

$\Rightarrow \quad -(xp^2+2yp+z)(xz-y^2) = 0$

$\therefore \quad$ Either $xp^2 + 2yp + z = 0$ or $y^2 = xz$

$\Rightarrow x, y, z$ are in GP.

21. Since, A is the determinant of order 3 with entries 0 or 1 only.

Also, B is the subset of A consisting of all determinants with value 1.

[since, if we interchange any two rows or columns, then among themself sign changes]

Given, C is the subset having determinant with value -1.

$\therefore$ B has as many elements as C.

22. For a matrix to be square of other matrix its determinant should be positive.

(a) and (c) $\to$ Correct

(b) and (d) $\to$ Incorrect

23. Given determinant could be expressed as product of two determinants.

i.e. $\begin{vmatrix} (1+\alpha)^2 & (1+2\alpha)^2 & (1+3\alpha)^2 \\ (2+\alpha)^2 & (2+2\alpha)^2 & (2+3\alpha)^2 \\ (3+\alpha)^2 & (3+2\alpha)^2 & (3+3\alpha)^2 \end{vmatrix} = -648\,\alpha$

$$\Rightarrow \begin{vmatrix} 1+2\alpha+\alpha^2 & 1+4\alpha+4\alpha^2 & 1+6\alpha+9\alpha^2 \\ 4+4\alpha+\alpha^2 & 4+8\alpha+4\alpha^2 & 4+12\alpha+9\alpha^2 \\ 9+6\alpha+\alpha^2 & 9+12\alpha+4\alpha^2 & 9+18\alpha+9\alpha^2 \end{vmatrix} = -648\,\alpha$$

$$\Rightarrow \begin{vmatrix} 1 & \alpha & \alpha^2 \\ 4 & 2\alpha & \alpha^2 \\ 9 & 3\alpha & \alpha^2 \end{vmatrix} \cdot \begin{vmatrix} 1 & 1 & 1 \\ 2 & 4 & 6 \\ 1 & 4 & 9 \end{vmatrix} = -648\,\alpha$$

$$\Rightarrow \alpha^3 \begin{vmatrix} 1 & 1 & 1 \\ 4 & 2 & 1 \\ 9 & 3 & 1 \end{vmatrix} \cdot \begin{vmatrix} 1 & 1 & 1 \\ 2 & 4 & 6 \\ 1 & 4 & 9 \end{vmatrix} = -648\,\alpha$$

$$\Rightarrow \quad -8\alpha^3 = -648\alpha$$

$$\Rightarrow \quad \alpha^3 - 81\alpha = 0 \Rightarrow \alpha(\alpha^2 - 81) = 0$$

$$\therefore \quad \alpha = 0, \pm 9$$

24. **PLAN** (i) If A and B are two non-zero matrices and $AB = BA$, then $(A-B)(A+B) = A^2 - B^2$.

(ii) The determinant of the product of the matrices is equal to product of their individual determinants, i.e. $|AB| = |A||B|$.

Given, $M^2 = N^4 \Rightarrow M^2 - N^4 = 0$

$\Rightarrow (M - N^2)(M + N^2) = 0$ [as $MN = NM$]

Also, $M \neq N^2$

$\Rightarrow M + N^2 = 0$

$\Rightarrow \det(M + N^2) = 0$

Also, $\det(M^2 + MN^2) = (\det M)(\det M + N^2)$

$= (\det M)(0) = 0$

As, $\det(M^2 + MN^2) = 0$

Thus, there exists a non-zero matrix U such that

$(M^2 + MN^2)U = 0$

25. Given, $\begin{vmatrix} a & b & a\alpha + b \\ b & c & b\alpha + c \\ a\alpha + b & b\alpha + c & 0 \end{vmatrix} = 0$

Applying $C_3 \to C_3 - (\alpha C_1 + C_2)$

$$\begin{vmatrix} a & b & 0 \\ b & c & 0 \\ a\alpha + b & b\alpha + c & -(a\alpha^2 + 2b\alpha + c) \end{vmatrix} = 0$$

$\Rightarrow -(a\alpha^2 + 2b\alpha + c)(ac - b^2) = 0$

$\Rightarrow a\alpha^2 + 2b\alpha + c = 0$ or $b^2 = ac$

$\Rightarrow x - \alpha$ is a factor of $ax^2 + 2bx + c$ or a, b, c are in GP.

26. Let $\text{Det}(P) = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$= a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$

Now, maximum value of Det $(P) = 6$

If $a_1 = 1, a_2 = -1, a_3 = 1, b_2c_3 = b_1c_3 = b_1c_2 = 1$

and $b_3c_2 = b_3c_1 = b_2c_1 = -1$

But it is not possible as

$(b_2c_3)(b_3c_1)(b_1c_2) = -1$ and $(b_1c_3)(b_3c_2)(b_2c_1) = 1$

i.e., $b_1b_2b_3c_1c_2c_3 = 1$ and -1

Similar contradiction occurs when

$a_1 = 1, a_2 = 1, a_3 = 1, b_2c_1 = b_3c_1 = b_1c_2 = 1$

and $b_3c_2 = b_1c_3 = b_1c_2 = -1$

Now, for value to be 5 one of the terms must be zero but that will make 2 terms zero which means answer cannot be 5

Now, $\begin{vmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{vmatrix} = 4$

Hence, maximum value is 4.

27. Let $\Delta = \begin{vmatrix} 1 & \log_x y & \log_x z \\ \log_y x & 1 & \log_y z \\ \log_z x & \log_z y & 1 \end{vmatrix}$

$$= \begin{vmatrix} 1 & \frac{\log y}{\log x} & \frac{\log z}{\log x} \\ \frac{\log x}{\log y} & 1 & \frac{\log z}{\log y} \\ \frac{\log x}{\log z} & \frac{\log y}{\log z} & 1 \end{vmatrix}$$

On dividing and multiplying R_1, R_2, R_3 by $\log x$, $\log y$, $\log z$, respectively.

$$= \frac{1}{\log x \log y \log z} \begin{vmatrix} \log x & \log y & \log z \\ \log x & \log y & \log z \\ \log x & \log y & \log z \end{vmatrix} = 0$$

28. $$\begin{vmatrix} 1 & a & a^2 - bc \\ 1 & b & b^2 - ca \\ 1 & c & c^2 - ab \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} - \begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$$

Now, $\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix} = \frac{1}{abc}\begin{vmatrix} a & a^2 & abc \\ b & b^2 & abc \\ c & c^2 & abc \end{vmatrix}$

Applying $R_1 \to aR_1, R_2 \to bR_2, R_3 \to cR_3$

$$= \frac{1}{abc} \cdot abc \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

$$\therefore \begin{vmatrix} 1 & a & a^2 - bc \\ 1 & b & b^2 - ca \\ 1 & c & c^2 - ab \end{vmatrix} = 0$$

29. Given, $\begin{vmatrix} x & 3 & 7 \\ 2 & x & 2 \\ 7 & 6 & x \end{vmatrix} = 0$

Applying $R_1 \to R_1 + R_2 + R_3$

$$\Rightarrow \begin{vmatrix} x+9 & x+9 & x+9 \\ 2 & x & 2 \\ 7 & 6 & x \end{vmatrix} = 0 \Rightarrow (x+9)\begin{vmatrix} 1 & 1 & 1 \\ 2 & x & 2 \\ 7 & 6 & x \end{vmatrix} = 0$$

Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$

$$\Rightarrow (x+9)\begin{vmatrix} 1 & 0 & 0 \\ 2 & x-2 & 0 \\ 7 & -1 & x-7 \end{vmatrix} = 0 \Rightarrow (x+9)(x-2)(x-7) = 0$$

$\Rightarrow$ $x = -9, 2, 7$ are the roots.

$\therefore$ Other two roots are 2 and 7.

30. Given, $\begin{vmatrix} 1 & 4 & 20 \\ 1 & -2 & 5 \\ 1 & 2x & 5x^2 \end{vmatrix} = 0$

$\Rightarrow$ $1(-10x^2 - 10x) - 4(5x^2 - 5) + 20(2x + 2) = 0$

$\Rightarrow$ $-30x^2 + 30x + 60 = 0$

$\Rightarrow$ $(x-2)(x+1) = 0$

$\Rightarrow$ $x = 2, -1$

Hence, the solution set is $\{-1, 2\}$.

31. Given, $\begin{vmatrix} \lambda^2+3\lambda & \lambda-1 & \lambda+3 \\ \lambda+1 & -2\lambda & \lambda-4 \\ \lambda-3 & \lambda+4 & 3\lambda \end{vmatrix}$

$$= p\lambda^4 + q\lambda^3 + r\lambda^2 + s\lambda + t$$

Thus, the value of t is obtained by putting $\lambda = 0$.

$$\Rightarrow \begin{vmatrix} 0 & -1 & 3 \\ 1 & 0 & -4 \\ -3 & 4 & 0 \end{vmatrix} = t$$

$\Rightarrow$ $t = 0$

[$\because$ determinants of odd order skew-symmetric matrix is zero]

32. Let $\Delta = \begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix} = \frac{1}{abc}\begin{vmatrix} a & a^2 & abc \\ b & b^2 & abc \\ c & c^2 & abc \end{vmatrix}$

Applying $R_1 \to aR_1, R_2 \to bR_2, R_3 \to cR_3$

$$= \frac{1}{abc} \cdot abc \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

$$\therefore \begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

Hence, statement is false.

33. Since, $M^T M = I$ and $|M| = 1$

$\therefore$ $|M - I| = |IM - M^T M|$ [$\because IM = M$]

$\Rightarrow |M - I| = |(I - M^T)M| = |(I - M)^T||M| = |I - M|$

$= (-1)^3 |M - I|$ [$\because I - M$ is a 3×3 matrix]

$= -|M - I|$

$\Rightarrow$ $2|M - I| = 0$

$\Rightarrow$ $|M - I| = 0$

34. Given, $\begin{vmatrix} ax-by-c & bx+ay & cx+a \\ bx+ay & -ax+by-c & cy+b \\ cx+a & cy+b & -ax-by+c \end{vmatrix} = 0$

$$\Rightarrow \frac{1}{a}\begin{vmatrix} a^2x-aby-ac & bx+ay & cx+a \\ abx+a^2y & -ax+by-c & cy+b \\ acx+a^2 & cy+b & -ax-by+c \end{vmatrix} = 0$$

Applying $C_1 \to C_1 + bC_2 + cC_3$

$$\Rightarrow \frac{1}{a}\begin{vmatrix} (a^2+b^2+c^2)x & ay+bx & cx+a \\ (a^2+b^2+c^2)y & by-c-ax & b+cy \\ a^2+b^2+c^2 & b+cy & c-ax-by \end{vmatrix} = 0$$

$$\Rightarrow \frac{1}{a}\begin{vmatrix} x & ay+bx & cx+a \\ y & by-c-ax & b+cy \\ 1 & b+cy & c-ax-by \end{vmatrix} = 0$$

[$\because a^2 + b^2 + c^2 = 1$]

Applying $C_2 \to C_2 - bC_1$ and $C_3 \to C_3 - cC_1$

$$\Rightarrow \frac{1}{a}\begin{vmatrix} x & ay & a \\ y & -c-ax & b \\ 1 & cy & -ax-by \end{vmatrix} = 0$$

$$\Rightarrow \frac{1}{ax}\begin{vmatrix} x^2 & axy & ax \\ y & -c-ax & b \\ 1 & cy & -ax-by \end{vmatrix} = 0$$

Applying $R_1 \to R_1 + yR_2 + R_3$

$$\Rightarrow \frac{1}{ax}\begin{vmatrix} x^2+y^2+1 & 0 & 0 \\ y & -c-ax & b \\ 1 & cy & -ax-by \end{vmatrix} = 0$$

$\Rightarrow$ $\frac{1}{ax}[(x^2 + y^2 + 1)\{(-c-ax)(-ax-by) - b(cy)\}] = 0$

$\Rightarrow \frac{1}{ax}[(x^2 + y^2 + 1)(acx + bcy + a^2x^2 + abxy - bcy)] = 0$

$\Rightarrow$ $\frac{1}{ax}[(x^2 + y^2 + 1)(acx + a^2x^2 + abxy)] = 0$

$\Rightarrow$ $\frac{1}{ax}[ax(x^2 + y^2 + 1)(c + ax + by)] = 0$

$\Rightarrow$ $(x^2 + y^2 + 1)(ax + by + c) = 0$

$\Rightarrow$ $ax + by + c = 0$

which represents a straight line.

35. Let $\Delta = \begin{vmatrix} \sin\theta & \cos\theta & \sin 2\theta \\ \sin\left(\theta + \frac{2\pi}{3}\right) & \cos\left(\theta + \frac{2\pi}{3}\right) & \sin\left(2\theta + \frac{4\pi}{3}\right) \\ \sin\left(\theta - \frac{2\pi}{3}\right) & \cos\left(\theta - \frac{2\pi}{3}\right) & \sin\left(2\theta - \frac{4\pi}{3}\right) \end{vmatrix}$

Applying $R_2 \to R_2 + R_3$

$$= \begin{vmatrix} \sin\theta & \cos\theta & \sin 2\theta \\ \sin\left(\theta + \frac{2\pi}{3}\right) + \sin\left(\theta - \frac{2\pi}{3}\right) & \cos\left(\theta + \frac{2\pi}{3}\right) + \cos\left(\theta - \frac{2\pi}{3}\right) & \sin\left(2\theta + \frac{4\pi}{3}\right) + \sin\left(2\theta - \frac{4\pi}{3}\right) \\ \sin\left(\theta - \frac{2\pi}{3}\right) & \cos\left(\theta - \frac{2\pi}{3}\right) & \sin\left(2\theta - \frac{4\pi}{3}\right) \end{vmatrix}$$

Now, $\sin\left(\theta + \frac{2\pi}{3}\right) + \sin\left(\theta - \frac{2\pi}{3}\right)$

$$= 2\sin\left(\frac{\theta+\frac{2\pi}{3}+\theta-\frac{2\pi}{3}}{2}\right)\cos\left(\frac{\theta+\frac{2\pi}{3}-\theta+\frac{2\pi}{3}}{2}\right)$$

$$= 2\sin\theta\cos\frac{2\pi}{3} = 2\sin\theta\cos\left(\pi-\frac{\pi}{3}\right)$$

$$= -2\sin\theta\cos\frac{\pi}{3} = -\sin\theta$$

and $\cos\left(\theta+\frac{2\pi}{3}\right)+\cos\left(\theta-\frac{2\pi}{3}\right)$

$$= 2\cos\left(\frac{\theta+\frac{2\pi}{3}+\theta-\frac{2\pi}{3}}{2}\right)\cos\left(\frac{\theta+\frac{2\pi}{3}-\theta+\frac{2\pi}{3}}{2}\right)$$

$$= 2\cos\theta\cos\left(\frac{2\pi}{3}\right) = 2\cos\theta\left(-\frac{1}{2}\right) = -\cos\theta$$

and $\sin\left(2\theta+\frac{4\pi}{3}\right)+\sin\left(2\theta-\frac{4\pi}{3}\right)$

$$= 2\sin\left(\frac{2\theta+\frac{4\pi}{3}+2\theta-\frac{4\pi}{3}}{2}\right)\cos\left(\frac{2\theta+\frac{4\pi}{3}-2\theta+\frac{4\pi}{3}}{2}\right)$$

$$= 2\sin 2\theta\cos\frac{4\pi}{3} = 2\sin 2\theta\cos\left(\pi+\frac{\pi}{3}\right)$$

$$= -2\sin 2\theta\cos\frac{\pi}{3} = -\sin 2\theta$$

$$\therefore \Delta = \begin{vmatrix} \sin\theta & \cos\theta & \sin 2\theta \\ -\sin\theta & -\cos\theta & -\sin 2\theta \\ \sin\left(\theta-\frac{2\pi}{3}\right) & \cos\left(\theta-\frac{2\pi}{3}\right) & \sin\left(2\theta-\frac{4\pi}{3}\right) \end{vmatrix} = 0$$

[since, R_1 and R_2 are proportional]

36. Given, $f'(x) = \begin{vmatrix} 2ax & 2ax-1 & 2ax+b+1 \\ b & b+1 & -1 \\ 2(ax+b) & 2ax+2b+1 & 2ax+b \end{vmatrix}$

Applying $R_3 \to R_3 - R_1 - 2R_2$, we get

$$f'(x) = \begin{vmatrix} 2ax & 2ax-1 & 2ax+b+1 \\ b & b+1 & -1 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 2ax & 2ax-1 \\ b & b+1 \end{vmatrix} = \begin{vmatrix} 2ax & -1 \\ b & 1 \end{vmatrix} \quad [C_2 \to C_2 - C_1]$$

$\Rightarrow \quad f'(x) = 2ax + b$

On integrating, we get $f(x) = ax^2 + bx + c$,

where c is an arbitrary constant.

Since, f has maximum at $x = 5/2$.

$\Rightarrow \quad f'(5/2) = 0 \Rightarrow 5a + b = 0$ ...(i)

Also, $\quad f(0) = 2 \Rightarrow c = 2$ and $f(1) = 1$

$\Rightarrow \quad a + b + c = 1$ ...(ii)

On solving Eqs. (i) and (ii) for $a, b,$ we get

$$a = \frac{1}{4}, b = -\frac{5}{4}$$

Thus, $\quad f(x) = \frac{1}{4}x^2 - \frac{5}{4}x + 2$

37. Since, a, b, c are pth, qth and rth terms of HP.

$\Rightarrow \quad \frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in an AP.

$$\Rightarrow \left.\begin{aligned} \frac{1}{a} &= A + (p-1)D \\ \frac{1}{b} &= A + (q-1)D \\ \frac{1}{c} &= A + (r-1)D \end{aligned}\right\} \quad \text{...(i)}$$

Let $\Delta = \begin{vmatrix} bc & ca & ab \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix} = abc\begin{vmatrix} \frac{1}{a} & \frac{1}{b} & \frac{1}{c} \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix}$ [from Eq. (i)]

$$= abc\begin{vmatrix} A+(p-1)D & A+(q-1)D & A+(r-1)D \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix}$$

Applying $R_1 \to R_1 - (A-D)R_3 - DR_2$

$$= abc\begin{vmatrix} 0 & 0 & 0 \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} bc & ca & ab \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix} = 0$$

38. Given, $a > 0, d > 0$ and let

$$\Delta = \begin{vmatrix} \frac{1}{a} & \frac{1}{a(a+d)} & \frac{1}{(a+d)(a+2d)} \\ \frac{1}{(a+d)} & \frac{1}{(a+d)(a+2d)} & \frac{1}{(a+2d)(a+3d)} \\ \frac{1}{(a+2d)} & \frac{1}{(a+2d)(a+3d)} & \frac{1}{(a+3d)(a+4d)} \end{vmatrix}$$

Taking $\frac{1}{a(a+d)(a+2d)}$ common from R_1,

$\frac{1}{(a+d)(a+2d)(a+3d)}$ from R_2,

$\frac{1}{(a+2d)(a+3d)(a+4d)}$ from R_3

$$\Rightarrow \Delta = \frac{1}{a(a+d)^2(a+2d)^3(a+3d)^2(a+4d)}\begin{vmatrix} (a+d)(a+2d) & (a+2d) & a \\ (a+2d)(a+3d) & (a+3d) & (a+d) \\ (a+3d)(a+4d) & (a+4d) & (a+2d) \end{vmatrix}$$

$$\Rightarrow \Delta = \frac{1}{a(a+d)^2(a+2d)^3(a+3d)^2(a+4d)}\Delta'$$

where, $\Delta' = \begin{vmatrix} (a+d)(a+2d) & (a+2d) & a \\ (a+2d)(a+3d) & (a+3d) & (a+d) \\ (a+3d)(a+4d) & (a+4d) & (a+2d) \end{vmatrix}$

Applying $R_2 \to R_2 - R_1, R_3 \to R_3 - R_2$

$$\Rightarrow \quad \Delta' = \begin{vmatrix} (a+d)(a+2d) & (a+2d) & a \\ (a+2d)(2d) & d & d \\ (a+3d)(2d) & d & d \end{vmatrix}$$

Applying $R_3 \to R_3 - R_2$

$$\Delta' = \begin{vmatrix} (a+d)(a+2d) & (a+2d) & a \\ (a+2d)2d & d & d \\ 2d^2 & 0 & 0 \end{vmatrix}$$

Expanding along R_3, we get

$$\Delta' = 2d^2 \begin{vmatrix} a+2d & a \\ d & d \end{vmatrix}$$

$$\Delta' = (2d^2)(d)(a+2d-a) = 4d^4$$

$$\therefore \quad \Delta = \frac{4d^4}{a(a+d)^2(a+2d)^3(a+3d)^2(a+4d)}$$

39. Let $\Delta = \begin{vmatrix} \cos(A-P) & \cos(A-Q) & \cos(A-R) \\ \cos(B-P) & \cos(B-Q) & \cos(B-R) \\ \cos(C-P) & \cos(C-Q) & \cos(C-R) \end{vmatrix}$

$$\Rightarrow \quad \Delta = \begin{vmatrix} \cos A\cos P + \sin A \sin P & \cos(A-Q) & \cos(A-R) \\ \cos B\cos P + \sin B \sin P & \cos(B-Q) & \cos(B-R) \\ \cos C\cos P + \sin C \sin P & \cos(C-Q) & \cos(C-R) \end{vmatrix}$$

$$\Rightarrow \quad \Delta = \begin{vmatrix} \cos A\cos P & \cos(A-Q) & \cos(A-R) \\ \cos B\cos P & \cos(B-Q) & \cos(B-R) \\ \cos C\cos P & \cos(C-Q) & \cos(C-R) \end{vmatrix} + \begin{vmatrix} \sin A\sin P & \cos(A-Q) & \cos(A-R) \\ \sin B\sin P & \cos(B-Q) & \cos(B-R) \\ \sin C\sin P & \cos(C-Q) & \cos(C-R) \end{vmatrix}$$

$$\Rightarrow \quad \Delta = \cos P \begin{vmatrix} \cos A & \cos(A-Q) & \cos(A-R) \\ \cos B & \cos(B-Q) & \cos(B-R) \\ \cos C & \cos(C-Q) & \cos(C-R) \end{vmatrix} + \sin P \begin{vmatrix} \sin A & \cos(A-Q) & \cos(A-R) \\ \sin B & \cos(B-Q) & \cos(B-R) \\ \sin C & \cos(C-Q) & \cos(C-R) \end{vmatrix}$$

Applying $C_2 \to C_2 - C_1 \cos Q, C_3 \to C_3 - C_1 \cos R$ in first determinant and $C_2 \to C_2 - C_1 \sin Q$ and in second determinant

$$\Rightarrow \quad \Delta = \cos P \begin{vmatrix} \cos A & \sin A \sin Q & \sin A \sin R \\ \cos B & \sin B \sin Q & \sin B \sin R \\ \cos C & \sin C \sin Q & \sin C \sin R \end{vmatrix} + \sin P \begin{vmatrix} \sin A & \cos A\cos Q & \cos A \cos R \\ \sin B & \cos B \cos Q & \cos B \cos R \\ \sin C & \cos C \cos Q & \cos C \cos R \end{vmatrix}$$

$$\Delta = \cos P \sin Q \sin R \begin{vmatrix} \cos A & \sin A & \sin A \\ \cos B & \sin B & \sin B \\ \cos C & \sin C & \sin C \end{vmatrix} + \sin P \cos Q \cos R \begin{vmatrix} \sin A & \cos A & \cos A \\ \sin B & \cos B & \cos B \\ \sin C & \cos C & \cos C \end{vmatrix}$$

$$\Delta = 0 + 0 = 0$$

40. Given, $D = \begin{vmatrix} n! & (n+1)! & (n+2)! \\ (n+1)! & (n+2)! & (n+3)! \\ (n+2)! & (n+3)! & (n+4)! \end{vmatrix}$

Taking $n!$, $(n+1)!$ and $(n+2)!$ common from R_1, R_2 and R_3, respectively.

$$\therefore \quad D = n!\,(n+1)!\,(n+2)! \begin{vmatrix} 1 & (n+1) & (n+1)(n+2) \\ 1 & (n+2) & (n+2)(n+3) \\ 1 & (n+3) & (n+3)(n+4) \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_2$, we get

$$D = n!(n+1)!(n+2)! \begin{vmatrix} 1 & (n+1) & (n+1)(n+2) \\ 0 & 1 & 2n+4 \\ 0 & 1 & 2n+6 \end{vmatrix}$$

Expanding along C_1, we get

$$D = (n!)(n+1)!(n+2)![(2n+6) - (2n+4)]$$

$$D = (n!)(n+1)!(n+2)!\,[2]$$

On dividing both side by $(n!)^3$

$$\Rightarrow \quad \frac{D}{(n!)^3} = \frac{(n!)(n!)(n+1)(n!)(n+1)(n+2)2}{(n!)^3}$$

$$\Rightarrow \quad \frac{D}{(n!)^3} = 2(n+1)(n+1)(n+2)$$

$$\Rightarrow \quad \frac{D}{(n!)^3} = 2(n^3 + 4n^2 + 5n + 2) = 2n(n^2 + 4n + 5) + 4$$

$$\Rightarrow \frac{D}{(n!)^3} - 4 = 2n(n^2 + 4n + 5)$$

which shows that $\left[\frac{D}{(n!)^3} - 4\right]$ is divisible by n.

41. Let $\Delta = \begin{vmatrix} p & b & c \\ a & q & c \\ a & b & r \end{vmatrix}$

Applying $R_1 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we get

$$\Delta = \begin{vmatrix} p & b & c \\ a-p & q-b & 0 \\ a-p & 0 & r-c \end{vmatrix}$$

$$= c \begin{vmatrix} a-p & q-b \\ a-p & 0 \end{vmatrix} + (r-c) \begin{vmatrix} p & b \\ a-p & q-b \end{vmatrix}$$

$$= -c(a-p)(q-b) + (r-c)[p(q-b) - b(a-p)]$$

$$= -c(a-p)(q-b) + p(r-c)(q-b) - b(r-c)(a-p)$$

Since, $\Delta = 0$

$$\Rightarrow -c(a-p)(q-b) + p(r-c)(q-b) - b(r-c)(a-p) = 0$$

$$\Rightarrow \quad \frac{c}{r-c} + \frac{p}{p-a} + \frac{b}{q-b} = 0$$

[on dividing both sides by $(a-p)(q-b)(r-c)$]

$$\Rightarrow \quad \frac{p}{p-a} + \frac{b}{q-b} + 1 + \frac{c}{r-c} + 1 = 2$$

$$\Rightarrow \quad \frac{p}{p-a} + \frac{q}{q-b} + \frac{r}{r-c} = 2$$

42. We know, $A28 = A \times 100 + 2 \times 10 + 8$

$3B9 = 3 \times 100 + B \times 10 + 9$

and $62C = 6 \times 100 + 2 \times 10 + C$

Since, $A28, 3B9$ and $62C$ are divisible by k, therefore there exist positive integers m_1, m_2 and m_3 such that,

$100 \times A + 10 \times 2 + 8 = m_1 k$, $100 \times 3 + 10 \times B + 9 = m_2 k$

and $100 \times 6 + 10 \times 2 + C = m_3 k$ ... (i)

$$\therefore \quad \Delta = \begin{vmatrix} A & 3 & 6 \\ 8 & 9 & C \\ 2 & B & 2 \end{vmatrix}$$

Applying $R_2 \to 100R_1 + 10R_3 + R_2$

$$\Rightarrow \quad \Delta = \begin{vmatrix} A & 3 & 6 \\ 100A+2\times 10+8 & 100\times 3+10\times B+9 & 100\times 6+10\times 2+C \\ 2 & B & 2 \end{vmatrix}$$

$$= \begin{vmatrix} A & 3 & 6 \\ A28 & 3B9 & 62C \\ 2 & B & 2 \end{vmatrix} \qquad \text{[from Eq. (i)]}$$

$$= \begin{vmatrix} A & 3 & 6 \\ m_1k & m_2k & m_3k \\ 2 & B & 2 \end{vmatrix} = k\begin{vmatrix} A & 3 & 6 \\ m_1 & m_2 & m_3 \\ 2 & B & 2 \end{vmatrix}$$

$$\therefore \quad \Delta = mk$$

Hence, determinant is divisible by k.

43. Given, $\Delta_a = \begin{vmatrix} a-1 & n & 6 \\ (a-1)^2 & 2n^2 & 4n-2 \\ (a-1)^3 & 3n^3 & 3n^2-3n \end{vmatrix}$

$$\therefore \quad \sum_{a=1}^{n} \Delta_a = \begin{vmatrix} \sum_{a=1}^{n}(a-1) & n & 6 \\ \sum_{a=1}^{n}(a-1)^2 & 2n^2 & 4n-2 \\ \sum_{a=1}^{n}(a-1)^3 & 3n^3 & 3n^2-3n \end{vmatrix}$$

$$= \begin{vmatrix} \dfrac{n(n-1)}{2} & n & 6 \\ \dfrac{n(n-1)(2n-1)}{6} & 2n^2 & 4n-2 \\ \dfrac{n^2(n-1)^2}{4} & 3n^3 & 3n^2-3n \end{vmatrix}$$

$$= \frac{n^2(n-1)}{2}\begin{vmatrix} 1 & 1 & 6 \\ \dfrac{(2n-1)}{3} & 2n & 4n-2 \\ \dfrac{n(n-1)}{2} & 3n^2 & 3n^2-3n \end{vmatrix}$$

$$= \frac{n^3(n-1)}{12}\begin{vmatrix} 1 & 1 & 6 \\ 2n-1 & 6n & 12n-6 \\ n-1 & 6n & 6n-6 \end{vmatrix}$$

Applying $C_3 \to C_3 - 6C_1$

$$= \frac{n^3(n-1)}{12}\begin{vmatrix} 1 & 1 & 0 \\ 2n-1 & 6n & 0 \\ n-1 & 6n & 0 \end{vmatrix} = 0$$

$$\Rightarrow \quad \sum_{a=1}^{n} \Delta_a = c \qquad [c = 0, \text{i.e. constant}]$$

44. Let $\Delta = \begin{vmatrix} {}^xC_r & {}^xC_{r+1} & {}^xC_{r+2} \\ {}^yC_r & {}^yC_{r+1} & {}^yC_{r+2} \\ {}^zC_r & {}^zC_{r+1} & {}^zC_{r+2} \end{vmatrix}$

Applying $C_3 \to C_3 + C_2$

$$\Delta = \begin{vmatrix} {}^xC_r & {}^xC_{r+1} & {}^{x+1}C_{r+2} \\ {}^yC_r & {}^yC_{r+1} & {}^{y+1}C_{r+2} \\ {}^zC_r & {}^zC_{r+1} & {}^{z+1}C_{r+2} \end{vmatrix}$$

$[\because {}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r]$

Applying $C_2 \to C_2 + C_1$

$$\Delta = \begin{vmatrix} {}^xC_r & {}^{x+1}C_{r+1} & {}^{x+1}C_{r+2} \\ {}^yC_r & {}^{y+1}C_{r+1} & {}^{y+1}C_{r+2} \\ {}^zC_r & {}^{z+1}C_{r+1} & {}^{z+1}C_{r+2} \end{vmatrix}$$

Applying $C_3 \to C_3 + C_2$

$$\Rightarrow \quad \Delta = \begin{vmatrix} {}^xC_r & {}^{x+1}C_{r+1} & {}^{x+2}C_{r+2} \\ {}^yC_r & {}^{y+1}C_{r+1} & {}^{y+2}C_{r+2} \\ {}^zC_r & {}^{z+1}C_{r+1} & {}^{z+2}C_{r+2} \end{vmatrix}$$

Hence proved.

45. Since, α is repeated root of $f(x) = 0$.

$\therefore \; f(x) = a\,(x-\alpha)^2, a \in \text{constant} (\neq 0)$

Let $\quad \phi(x) = \begin{vmatrix} A(x) & B(x) & C(x) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix}$

To show $\phi(x)$ is divisible by $(x-\alpha)^2$, it is sufficient to show that $\phi(\alpha)$ and $\phi'(\alpha) = 0$.

$$\therefore \quad \phi(\alpha) = \begin{vmatrix} A(\alpha) & B(\alpha) & C(\alpha) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix}$$

$= 0 \qquad [\because R_1$ and R_2 are identical$]$

Again, $\phi'(x) = \begin{vmatrix} A'(x) & B'(x) & C'(x) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix}$

$$\phi'(\alpha) = \begin{vmatrix} A'(\alpha) & B'(\alpha) & C'(\alpha) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix}$$

$= 0 \qquad [\because R_1$ and R_3 are identical$]$

Thus, α is a repeated root of $\phi(x) = 0$.

Hence, $\phi(x)$ is divisible by $f(x)$.

46. Let $\Delta = \begin{vmatrix} x^2+x & x+1 & x-2 \\ 2x^2+3x-1 & 3x & 3x-3 \\ x^2+2x+3 & 2x-1 & 2x-1 \end{vmatrix}$

Applying $R_2 \to R_2 - (R_1 + R_3)$, we get

$$\Delta = \begin{vmatrix} x^2+x & x+1 & x-2 \\ -4 & 0 & 0 \\ x^2+2x+3 & 2x-1 & 2x-1 \end{vmatrix}$$

Applying $R_1 \to R_1 + \dfrac{x^2}{4}R_2$

and $R_3 \to R_3 + \dfrac{x^2}{4}R_2$, we get

$$\Delta = \begin{vmatrix} x & x+1 & x-2 \\ -4 & 0 & 0 \\ 2x+3 & 2x-1 & 2x-1 \end{vmatrix}$$

Applying $R_3 \to R_3 - 2R_1 = \begin{vmatrix} x+0 & x+1 & x-2 \\ -4 & 0 & 0 \\ 3 & -3 & 3 \end{vmatrix}$

$$= \begin{vmatrix} x & x & x \\ -4 & 0 & 0 \\ 3 & -3 & 3 \end{vmatrix} + \begin{vmatrix} 0 & 1 & -2 \\ -4 & 0 & 0 \\ 3 & -3 & 3 \end{vmatrix}$$

$$= x\begin{vmatrix} 1 & 1 & 1 \\ -4 & 0 & 0 \\ 3 & -3 & 3 \end{vmatrix} + \begin{vmatrix} 0 & 1 & -2 \\ -4 & 0 & 0 \\ 3 & -3 & 3 \end{vmatrix}$$

$\Rightarrow \quad \Delta = Ax + B$

where, $A = \begin{vmatrix} 1 & 1 & 1 \\ -4 & 0 & 0 \\ 3 & -3 & 3 \end{vmatrix}$

and $\quad B = \begin{vmatrix} 0 & 1 & -2 \\ -4 & 0 & 0 \\ 3 & -3 & 3 \end{vmatrix}$

47. Let $\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

Applying $C_1 \to C_1 + C_2 + C_3$

$$\Delta = \begin{vmatrix} a+b+c & b & c \\ a+b+c & c & a \\ a+b+c & a & b \end{vmatrix} = (a+b+c)\begin{vmatrix} 1 & b & c \\ 1 & c & a \\ 1 & a & b \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we get

$$= (a+b+c)\begin{vmatrix} 1 & b & c \\ 0 & c-b & a-c \\ 0 & a-b & b-c \end{vmatrix}$$

$$= (a+b+c)\,[-(c-b)^2 - (a-b)(a-c)]$$

$$= -(a+b+c)(a^2+b^2+c^2-ab-bc-ca)$$

$$= -\frac{1}{2}(a+b+c)(2a^2+2b^2+2c^2-2ab-2bc-2ca)$$

$$= -\frac{1}{2}(a+b+c)[(a-b)^2+(b-c)^2+(c-a)^2]$$

which is always negative.

48. Given, $\begin{vmatrix} x & x^2 & 1+x^3 \\ 2x & 4x^2 & 1+8x^3 \\ 3x & 9x^2 & 1+27x^3 \end{vmatrix} = 10$

$$\Rightarrow \quad x\cdot x^2\begin{vmatrix} 1 & 1 & 1+x^3 \\ 2 & 4 & 1+8x^3 \\ 3 & 9 & 1+27x^3 \end{vmatrix} = 10$$

Apply $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - 3R_1$, we get

$$x^3\begin{vmatrix} 1 & 1 & 1+x^3 \\ 0 & 2 & -1+6x^3 \\ 0 & 6 & -2+24x^3 \end{vmatrix} = 10$$

$$\Rightarrow \quad x^3\cdot\begin{vmatrix} 2 & 6x^3-1 \\ 6 & 24x^3-2 \end{vmatrix} = 10$$

$\Rightarrow \quad x^3(48x^3 - 4 - 36x^3 + 6) = 10$

$\Rightarrow \quad 12x^6 + 2x^3 = 10$

$\Rightarrow \quad 6x^6 + x^3 - 5 = 0$

$\Rightarrow \quad x^3 = \frac{5}{6}, -1$

$$x = \left(\frac{5}{6}\right)^{1/3}, -1$$

Hence, the number of real solutions is 2.