10.3.6 *Normal form* Suppose a non-vertical line is known to us with following data:

(i) Length of the perpendicular (normal) from origin to the line.

(ii) Angle which normal makes with the positive direction of x-axis.

Let L be the line, whose perpendicular distance from origin O be OA = p and the angle between the positive x-axis and OA be $\angle XOA = \omega$. The possible positions of line L in the Cartesian plane are shown in the Fig 10.17. Now, our purpose is to find slope of L and a point on it. Draw perpendicular AM on the x-axis in each case.

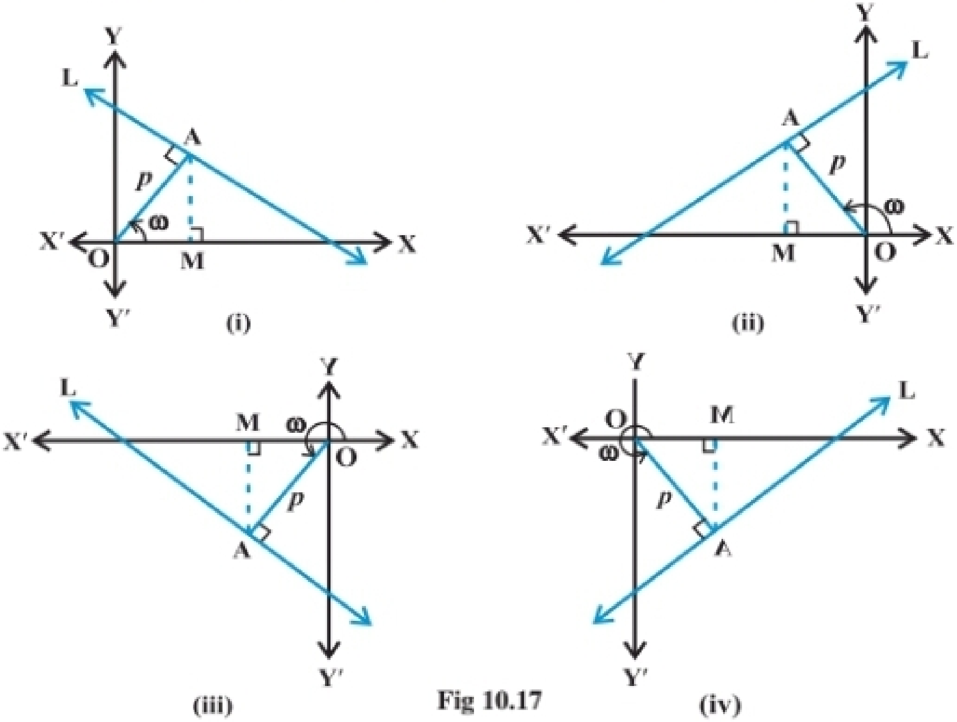

Fig 10.17

In each case, we have OM = $p \cos \omega$ and MA = $p \sin \omega$, so that the coordinates of the point A are $(p \cos \omega, p \sin \omega)$.

Further, line L is perpendicular to OA. Therefore

$$\text{The slope of the line L} = -\frac{1}{\text{slope of OA}} = -\frac{1}{\tan \omega} = -\frac{\cos \omega}{\sin \omega}.$$

Thus, the line L has slope $-\frac{\cos\omega}{\sin\omega}$ and point $A(p\cos\omega, p\sin\omega)$ on it. Therefore, by point-slope form, the equation of the line L is

$$y - p\sin\omega = -\frac{\cos\omega}{\sin\omega}(x - p\cos\omega) \quad \text{or} \quad x\cos\omega + y\sin\omega = p(\sin^2\omega + \cos^2\omega)$$

or $$x \cos \omega + y \sin \omega = p.$$

Hence, the equation of the line having normal distance p from the origin and angle ω which the normal makes with the positive direction of x-axis is given by

$$x \cos \omega + y \sin \omega = p \qquad \text{... (6)}$$

10.3.4 *Slope-intercept form* Sometimes a line is known to us with its slope and an intercept on one of the axes. We will now find equations of such lines.

Case I Suppose a line L with slope m cuts the y-axis at a distance c from the origin (Fig10.15). The distance c is called the *y-intercept* of the line L. Obviously, coordinates of the point where the line meet the y-axis are $(0, c)$. Thus, L has slope m and passes through a fixed point $(0, c)$. Therefore, by point-slope form, the equation of L is

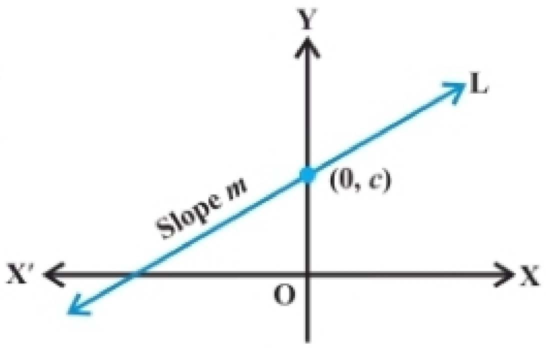

Fig 10.15

$$y - c = m(x - 0) \text{ or } y = mx + c$$

Thus, the point (x, y) on the line with slope m and y-intercept c lies on the line if and only if

$$y = mx + c \qquad ...(3)$$

Note that the value of c will be positive or negative according as the intercept is made on the positive or negative side of the y-axis, respectively.

Case II Suppose line L with slope m makes x-intercept d. Then equation of L is

$$y = m(x - d) \qquad ... (4)$$

Students may derive this equation themselves by the same method as in Case I.

Example 9 Write the equation of the lines for which $\tan \theta = \frac{1}{2}$, where θ is the inclination of the line and (i) y-intercept is $-\frac{3}{2}$ (ii) x-intercept is 4.

Solution (i) Here, slope of the line is $m = \tan \theta = \frac{1}{2}$ and y - intercept $c = -\frac{3}{2}$.

Therefore, by slope-intercept form (3) above, the equation of the line is

$$y = \frac{1}{2}x - \frac{3}{2} \text{ or } 2y - x + 3 = 0,$$

which is the required equation.

(ii) Here, we have $m = \tan \theta = \frac{1}{2}$ and $d = 4$.

Therefore, by slope-intercept form (4) above, the equation of the line is

$$y = \frac{1}{2}(x - 4) \text{ or } 2y - x + 4 = 0,$$

which is the required equation.

10.3.2 *Point-slope form* Suppose that P_0 (x_0, y_0) is a fixed point on a non-vertical line L, whose slope is m. Let P (x, y) be an arbitrary point on L (Fig 10.13).

Then, by the definition, the slope of L is given by

$$m = \frac{y-y}{x-x_0}, \text{ i.e., } y - y_0 = m(x - x_0) \qquad ...(1)$$

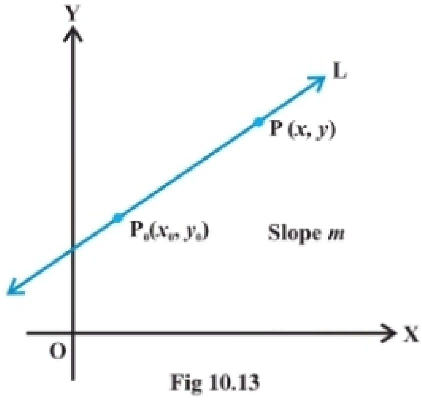

Fig 10.13

Since the point P_0 (x_0, y_0) along with all points (x, y) on L satisfies (1) and no other point in the plane satisfies (1). Equation (1) is indeed the equation for the given line L.

Thus, the point (x, y) lies on the line with slope m through the fixed point (x_0, y_0), if and only if, its coordinates satisfy the equation

$$y - y_0 = m(x - x_0)$$

Example 7 Find the equation of the line through (– 2, 3) with slope – 4.

Solution Here $m = -4$ and given point (x_0, y_0) is $(-2, 3)$.

By slope-intercept form formula (1) above, equation of the given line is

$y - 3 = -4(x + 2)$ or $4x + y + 5 = 0$, which is the required equation.

10.3.3 *Two-point form* Let the line L passes through two given points P_1 (x_1, y_1) and P_2 (x_2, y_2). Let P (x, y) be a general point on L (Fig 10.14).

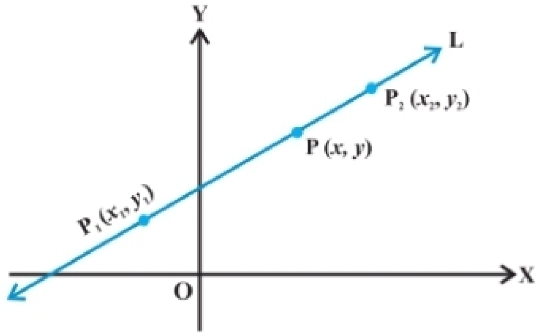

Fig 10.14

The three points P_1, P_2 and P are collinear, therefore, we have

slope of P_1P = slope of P_1P_2

i.e., $$\frac{y-y_1}{x-x_1} = \frac{y_2-y_1}{x_2-x_1}, \quad \text{or} \quad y - y_1 = \frac{y_2-y_1}{x_2-x_1}(x - x_1).$$

Thus, equation of the line passing through the points (x_1, y_1) and (x_2, y_2) is given by

$$y - y_1 = \frac{y_2-y_1}{x_2-x_1}(x - x_1) \qquad ... (2)$$

Example 8 Write the equation of the line through the points (1, –1) and (3, 5).

Solution Here $x_1 = 1$, $y_1 = -1$, $x_2 = 3$ and $y_2 = 5$. Using two-point form (2) above for the equation of the line, we have

$$y - (-1) = \frac{5-(-1)}{3-1}(x - 1)$$

or $\qquad -3x + y + 4 = 0$, which is the required equation.

10.2.3 ***Angle between two lines*** When we think about more than one line in a plane, then we find that these lines are either intersecting or parallel. Here we will discuss the angle between two lines in terms of their slopes.

Let L_1 and L_2 be two non-vertical lines with slopes m_1 and m_2, respectively. If α_1 and α_2 are the inclinations of lines L_1 and L_2, respectively. Then

$$m_1 = \tan\alpha_1 \text{ and } m_2 = \tan\alpha_2 .$$

We know that when two lines intersect each other, they make two pairs of vertically opposite angles such that sum of any two adjacent angles is 180°. Let θ and ϕ be the adjacent angles between the lines L_1 and L_2 (Fig10.6). Then

$$\theta = \alpha_2 - \alpha_1 \text{ and } \alpha_1, \alpha_2 \neq 90°.$$

Therefore $\tan\theta = \tan(\alpha_2 - \alpha_1) = \dfrac{\tan\alpha_2 - \tan\alpha_1}{1+\tan\alpha_1\tan\alpha_2} = \dfrac{m_2 - m_1}{1+m_1m_2}$ (as $1 + m_1m_2 \neq 0$)

and $\phi = 180° - \theta$ so that

$$\tan\phi = \tan(180° - \theta) = -\tan\theta = -\frac{m_2-m_1}{1+m_1m_2}, \text{ as } 1 + m_1m_2 \neq 0$$

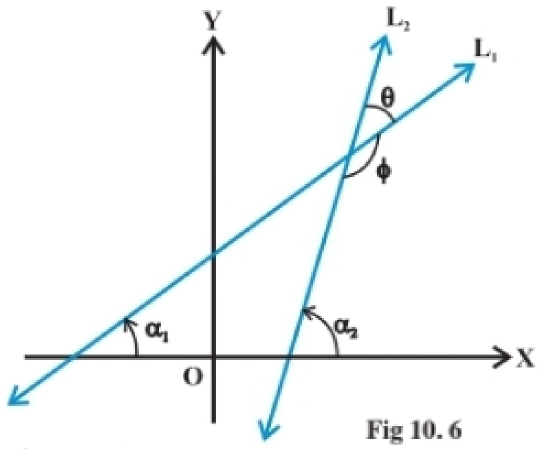

Fig 10. 6

Now, there arise two cases:

Case I If $\dfrac{m_2 - m_1}{1 + m_1m_2}$ is positive, then tan θ will be positive and tan ϕ will be negative, which means θ will be acute and ϕ will be obtuse.

Case II If $\dfrac{m_2 - m_1}{1 + m_1m_2}$ is negative, then tan θ will be negative and tan ϕ will be positive, which means that θ will be obtuse and ϕ will be acute.

Thus, the acute angle (say θ) between lines L_1 and L_2 with slopes m_1 and m_2, respectively, is given by

$$\tan\theta = \left|\frac{m_2 - m_1}{1+m_1m_2}\right|, \text{ as } 1+m_1m_2 \neq 0 \quad \text{... (1)}$$

The obtuse angle (say ϕ) can be found by using $\phi = 180^0 - \theta$.

10.2.2 ***Conditions for parallelism and perpendicularity of lines in terms of their slopes*** In a coordinate plane, suppose that non-vertical *lines* l_1 and l_2 have slopes m_1 *and* m_2, respectively. Let their inclinations be α and β, respectively.

If the line l_1 is parallel to l_2 (Fig 10.4), then their inclinations are equal, i.e.,

$$\alpha = \beta, \text{ and hence, } \tan\alpha = \tan\beta$$

Therefore $m_1 = m_2$, i.e., their slopes are equal.

Conversely, if the slope of two lines l_1 and l_2 is same, i.e.,

$$m_1 = m_2.$$

Then $\tan\alpha = \tan\beta$.

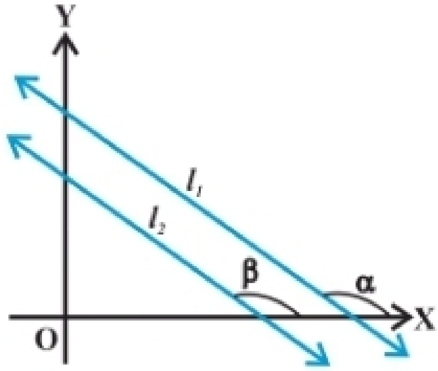

Fig 10. 4

By the property of tangent function (between 0° and 180°), α = β.

Therefore, the lines are parallel.

Hence, two non vertical lines l_1 and l_2 are parallel if and only if their slopes are equal.

If the lines l_1 and l_2 are perpendicular (Fig 10.5), then $\beta = \alpha + 90°$.

Therefore, $\tan\beta = \tan(\alpha + 90°)$

$$= -\cot\alpha = -\frac{1}{\tan\alpha}$$

i.e., $m_2 = -\frac{1}{m_1}$ or $m_1\, m_2 = -1$

Conversely, if $m_1\, m_2 = -1$, i.e., $\tan\alpha\tan\beta = -1$.

Then $\tan\alpha = -\cot\beta = \tan(\beta + 90°)$ or $\tan(\beta - 90°)$

Therefore, α and β differ by 90°.

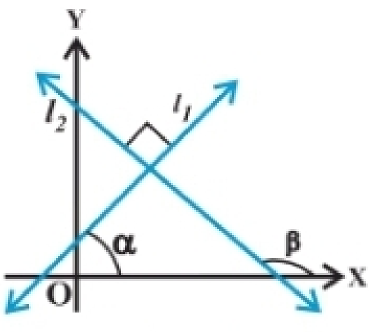

Fig 10. 5

Thus, lines l_1 and l_2 are perpendicular to each other.

Hence, *two non-vertical lines are perpendicular to each other if and only if their slopes are negative reciprocals of each other,*

i.e., $m_2 = -\frac{1}{m_1}$ or, $m_1\, m_2 = -1$.

Let us consider the following example.