1.15 Applications of Gauss's Law

The electric field due to a general charge distribution is, as seen above, given by Eq. (1.27). In practice, except for some special cases, the summation (or integration) involved in this equation cannot be carried out to give electric field at every point in space. For some symmetric charge configurations, however, it is possible to obtain the electric field in a simple way using the Gauss's law. This is best understood by some examples.

1.15.1 Field due to an infinitely long straight uniformly charged wire

Consider an infinitely long thin straight wire with uniform linear charge density λ. The wire is obviously an axis of symmetry. Suppose we take the radial vector from O to P and rotate it around the wire. The points P, P′, P″ so obtained are completely equivalent with respect to the charged wire. This implies that the electric field must have the same magnitude at these points. The direction of electric field at every point must be radial (outward if $\lambda > 0$, inward if $\lambda < 0$). This is clear from Fig. 1.29.

Consider a pair of line elements P_1 and P_2 of the wire, as shown. The electric fields produced by the two elements of the pair when summed give a resultant electric field which is radial (the components normal to the radial vector cancel). This is true for any such pair and hence the total field at any point P is radial. Finally, since the wire is infinite, electric field does not depend on the position of P along the length of the wire. In short, the electric field is everywhere radial in the plane cutting the wire normally, and its magnitude depends only on the radial distance r.

To calculate the field, imagine a cylindrical Gaussian surface, as shown in the Fig. 1.29(b). Since the field is everywhere radial, flux through the two ends of the cylindrical Gaussian surface is zero. At the cylindrical part of the surface, **E** is normal to the surface at every point, and its magnitude is constant, since it depends only on r. The surface area of the curved part is $2\pi rl$, where l is the length of the cylinder.

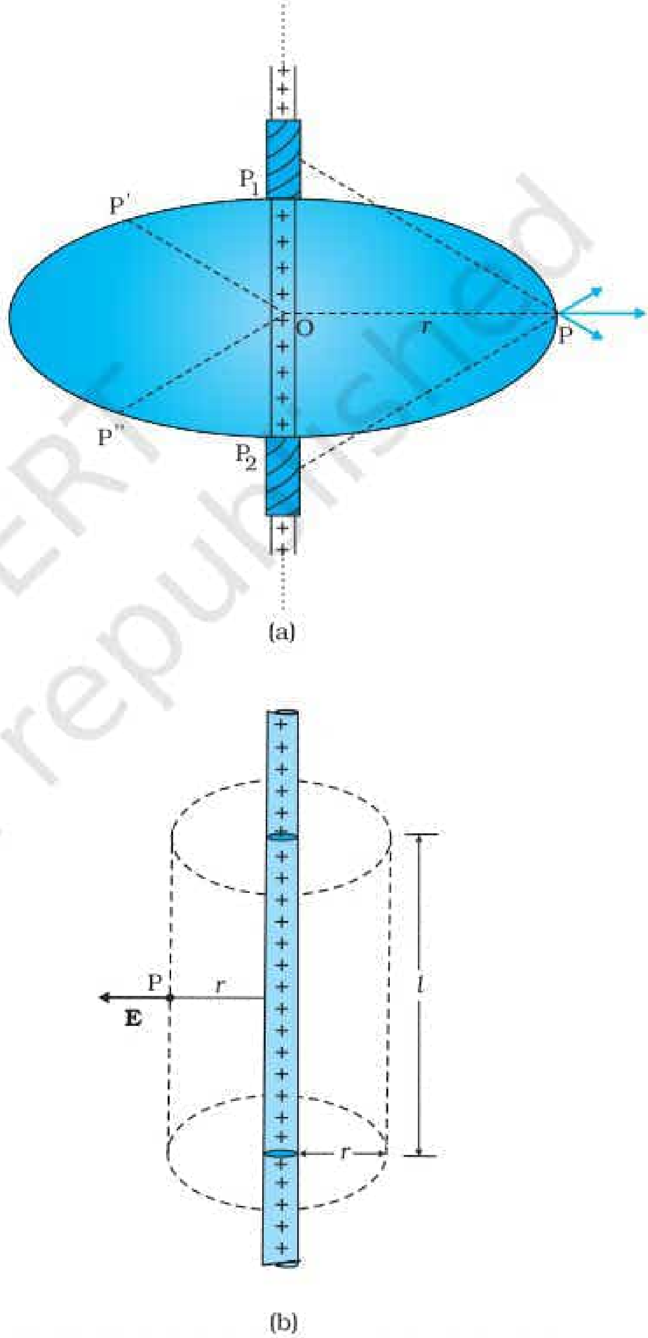

FIGURE 1.29 (a) Electric field due to an infinitely long thin straight wire is radial, (b) The Gaussian surface for a long thin wire of uniform linear charge density.

Flux through the Gaussian surface

= flux through the curved cylindrical part of the surface

= $E \times 2\pi rl$

The surface includes charge equal to λl. Gauss's law then gives $E \times 2\pi rl = \lambda l/\varepsilon_0$

i.e., $E = \dfrac{\lambda}{2\pi\varepsilon_0 r}$

Vectorially, **E** at any point is given by

$$\mathbf{E} = \frac{\lambda}{2\pi\varepsilon_0 r}\,\hat{\mathbf{n}} \qquad (1.32)$$

where $\hat{\mathbf{n}}$ is the radial unit vector in the plane normal to the wire passing through the point. **E** is directed outward if λ is positive and inward if λ is negative.

Note that when we write a vector **A** as a scalar multiplied by a unit vector, i.e., as $\mathbf{A} = A\,\hat{\mathbf{a}}$, the scalar A is an algebraic number. It can be negative or positive. The direction of **A** will be the same as that of the unit vector $\hat{\mathbf{a}}$ if $A > 0$ and opposite to $\hat{\mathbf{a}}$ if $A < 0$. When we want to restrict to non-negative values, we use the symbol $|\mathbf{A}|$ and call it the modulus of **A**. Thus, $|\mathbf{A}| \geq 0$.

Also note that though only the charge enclosed by the surface (λl) was included above, the electric field **E** is due to the charge on the entire wire. Further, the assumption that the wire is infinitely long is crucial. Without this assumption, we cannot take **E** to be normal to the curved part of the cylindrical Gaussian surface. However, Eq. (1.32) is approximately true for electric field around the central portions of a long wire, where the end effects may be ignored.

1.15.2 Field due to a uniformly charged infinite plane sheet

Let σ be the uniform surface charge density of an infinite plane sheet (Fig. 1.30). We take the x-axis normal to the given plane. By symmetry, the electric field will not depend on y and z coordinates and its direction at every point must be parallel to the x-direction.

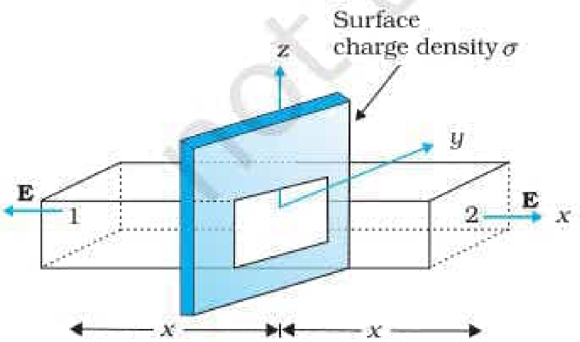

FIGURE 1.30 Gaussian surface for a uniformly charged infinite plane sheet.

We can take the Gaussian surface to be a rectangular parallelepiped of cross-sectional area A, as shown. (A cylindrical surface will also do.) As seen from the figure, only the two faces 1 and 2 will contribute to the flux; electric field lines are parallel to the other faces and they, therefore, do not contribute to the total flux.

The unit vector normal to surface 1 is in $-x$ direction while the unit vector normal to surface 2 is in the $+x$ direction. Therefore, flux $\mathbf{E}.\Delta\mathbf{S}$ through both the surfaces are equal and add up. Therefore the net flux through the Gaussian surface is $2\,EA$. The charge enclosed by the closed surface is σA. Therefore by Gauss's law,

$2\,EA = \sigma A/\varepsilon_0$
or, $E = \sigma/2\varepsilon_0$
Vectorically,

$$\mathbf{E} = \frac{\sigma}{2\varepsilon_0}\hat{\mathbf{n}} \tag{1.33}$$

where $\hat{\mathbf{n}}$ is a unit vector normal to the plane and going away from it.

E is directed away from the plate if σ is positive and toward the plate if σ is negative. Note that the above application of the Gauss' law has brought out an additional fact: E is independent of x also.

For a finite large planar sheet, Eq. (1.33) is approximately true in the middle regions of the planar sheet, away from the ends.

1.15.3 Field due to a uniformly charged thin spherical shell

Let σ be the uniform surface charge density of a thin spherical shell of radius R (Fig. 1.31). The situation has obvious spherical symmetry. The field at any point P, outside or inside, can depend only on r (the radial distance from the centre of the shell to the point) and must be radial (i.e., along the radius vector).

(i) Field outside the shell: Consider a point P outside the shell with radius vector **r**. To calculate **E** at P, we take the Gaussian surface to be a sphere of radius r and with centre O, passing through P. All points on this sphere are equivalent relative to the given charged configuration. (That is what we mean by spherical symmetry.) The electric field at each point of the Gaussian surface, therefore, has the same magnitude E and is along the radius vector at each point. Thus, **E** and $\Delta\mathbf{S}$ at every point are parallel and the flux through each element is $E\,\Delta S$. Summing over all ΔS, the flux through the Gaussian surface is $E \times 4\,\pi\, r^2$. The charge enclosed is $\sigma \times 4\,\pi\, R^2$. By Gauss's law

$$E \times 4\,\pi\, r^2 = \frac{\sigma}{\varepsilon_0}\,4\,\pi\, R^2$$

$$\text{Or,}\quad E = \frac{\sigma R^2}{\varepsilon_0\, r^2} = \frac{q}{4\pi\varepsilon_0 r^2}$$

where $q = 4\,\pi R^2\,\sigma$ is the total charge on the spherical shell.
Vectorially,

$$\mathbf{E} = \frac{q}{4\pi\varepsilon_0\, r^2}\hat{\mathbf{r}} \tag{1.34}$$

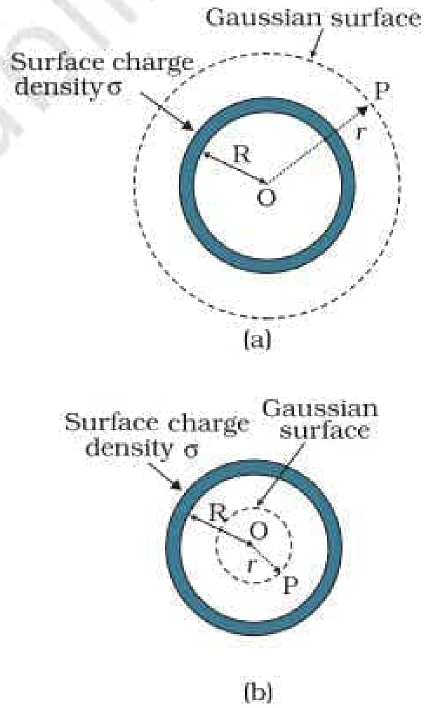

FIGURE 1.31 Gaussian surfaces for a point with (a) $r > R$, (b) $r < R$.

The electric field is directed outward if $q > 0$ and inward if $q < 0$. This, however, is exactly the field produced by a charge q placed at the centre O. Thus for points outside the shell, the field due to a uniformly charged shell is as if the entire charge of the shell is concentrated at its centre.

(ii) Field inside the shell: In Fig. 1.31(b), the point P is inside the shell. The Gaussian surface is again a sphere through P centred at O.

The flux through the Gaussian surface, calculated as before, is $E \times 4\,\pi\,r^2$. However, in this case, the Gaussian surface encloses no charge. Gauss's law then gives

$E \times 4\,\pi\,r^2 = 0$

i.e., $E = 0 \qquad (r < R)$ (1.35)

that is, the field due to a uniformly charged thin shell is zero at all points inside the shell*. This important result is a direct consequence of Gauss's law which follows from Coulomb's law. The experimental verification of this result confirms the $1/r^2$ dependence in Coulomb's law.

Example 1.13 An early model for an atom considered it to have a positively charged point nucleus of charge *Ze*, surrounded by a uniform density of negative charge up to a radius *R*. The atom as a whole is neutral. For this model, what is the electric field at a distance *r* from the nucleus?

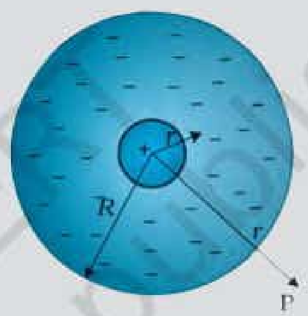

FIGURE 1.32

Solution The charge distribution for this model of the atom is as shown in Fig. 1.32. The total negative charge in the uniform spherical charge distribution of radius *R* must be $-Z\,e$, since the atom (nucleus of charge $Z\,e$ + negative charge) is neutral. This immediately gives us the negative charge density ρ, since we must have

$$\frac{4\pi R^3}{3}\rho = 0 - Ze$$

or $\rho = -\dfrac{3Ze}{4\pi R^3}$

To find the electric field **E**(**r**) at a point P which is a distance *r* away from the nucleus, we use Gauss's law. Because of the spherical symmetry of the charge distribution, the magnitude of the electric field **E**(**r**) depends only on the radial distance, no matter what the direction of **r**. Its direction is along (or opposite to) the radius vector **r** from the origin to the point P. The obvious Gaussian surface is a spherical surface centred at the nucleus. We consider two situations, namely, $r < R$ and $r > R$.

(i) $r < R$: The electric flux ϕ enclosed by the spherical surface is

$$\phi = E\,(r) \times 4\,\pi\,r^2$$

where $E\,(r)$ is the magnitude of the electric field at *r*. This is because

EXAMPLE 1.13

* Compare this with a uniform mass shell discussed in Section 8.5 of Class XI Textbook of Physics.

the field at any point on the spherical Gaussian surface has the same direction as the normal to the surface there, and has the same magnitude at all points on the surface.

The charge q enclosed by the Gaussian surface is the positive nuclear charge and the negative charge within the sphere of radius r,

i.e., $q = Ze + \frac{4\pi r^3}{3}\rho$

Substituting for the charge density ρ obtained earlier, we have

$$q = Ze - Ze\frac{r^3}{R^3}$$

Gauss's law then gives,

$$E(r) = \frac{Ze}{4\pi\varepsilon_0}\;\frac{1}{r^2} - \frac{r}{R^3} \;; \quad r < R$$

The electric field is directed radially outward.

(ii) $r > R$: In this case, the total charge enclosed by the Gaussian spherical surface is zero since the atom is neutral. Thus, from Gauss's law,

$E(r) \times 4\pi\ r^2 = 0$ or $E(r) = 0$; $\quad r > R$

At $r = R$, both cases give the same result: $E = 0$.

EXAMPLE 1.13