Subjective - 1998

Q: Nuclei of a radioactive element A are being produced at a constant rate α. The element has a decay constant λ. At time $t=0$, there are N_0 nuclei of the element.

a) Calculate the number N of nuclei of A at time t.

b) If $\alpha = 2N_0\lambda$, calculate the number of number of A after one half-life of A, and also the limiting value of N as $t \to \infty$.

A: a) $\xrightarrow{\alpha}$ (A) $\xrightarrow{\lambda N}$

Rate of production $= \alpha$

Rate of decay $= \lambda N$

Net rate of production $= \alpha - \lambda N$

$$\frac{dN}{dt} = \alpha - \lambda N$$

$$\int_{N_0}^{N} \frac{dN}{\alpha - \lambda N} = \int_0^t dt$$

$$\left[\frac{\ln(\alpha - \lambda N)}{-\lambda}\right]_{N_0}^{N} = (t - 0)$$

$$\ln\left(\frac{\alpha - \lambda N}{\alpha - \lambda N_0}\right) = -\lambda t$$

$$(\alpha - \lambda N) = (\alpha - \lambda N_0)e^{-\lambda t}$$

$$\lambda N = \alpha - (\alpha - \lambda N_0)e^{-\lambda t}$$

$$N(t) = \frac{1}{\lambda}\left[\alpha - (\alpha - \lambda N_0)e^{-\lambda t}\right]$$

(b) $t_{1/2} = \frac{\ln 2}{\lambda}$ $\qquad \alpha = 2\lambda N_0$

$$N(t) = \frac{1}{\lambda}\left[2N_0\lambda - N_0\lambda e^{-\lambda t}\right]$$

At $t = \frac{\ln 2}{\lambda}$

$$N(t = t_{1/2}) = \frac{1}{\lambda}\left[2N_0\lambda - \frac{N_0\lambda}{2}\right] = \frac{1}{\lambda}\left(\frac{3}{2}N_0\lambda\right) = \frac{3N_0}{2}$$

As $t \to \infty$, $e^{-\lambda t} \longrightarrow 0$

$$\Rightarrow \lim_{t\to\infty} N(t) = \frac{1}{\lambda}\left(2N_0\lambda - 0\right) = 2N_0$$

$\left(\text{Ans} = \frac{3N_0}{2}, 2N_0\right)$