EXERCISE 8.2

Find the coefficient of

1. x^5 in $(x + 3)^8$ **2.** a^5b^7 in $(a - 2b)^{12}$.

Write the general term in the expansion of

3. $(x^2 - y)^6$ **4.** $(x^2 - yx)^{12}, x \neq 0$.

5. Find the 4^{th} term in the expansion of $(x - 2y)^{12}$.

6. Find the 13^{th} term in the expansion of $\left(9x - \frac{1}{3\sqrt{x}}\right)^{18}$, $x \neq 0$.

Find the middle terms in the expansions of

7. $\left(3 - \frac{x^3}{6}\right)^7$ **8.** $\left(\frac{x}{3} + 9y\right)^{10}$.

9. In the expansion of $(1 + a)^{m+n}$, prove that coefficients of a^m and a^n are equal.

10. The coefficients of the $(r - 1)^{th}$, r^{th} and $(r + 1)^{th}$ terms in the expansion of $(x + 1)^n$ are in the ratio 1 : 3 : 5. Find n and r.

11. Prove that the coefficient of x^n in the expansion of $(1 + x)^{2n}$ is twice the coefficient of x^n in the expansion of $(1 + x)^{2n-1}$.

12. Find a positive value of m for which the coefficient of x^2 in the expansion $(1 + x)^m$ is 6.

Miscellaneous Examples

Example 10 Find the term independent of x in the expansion of $\left(\frac{3}{2}x^2 - \frac{1}{3x}\right)^6$.

Solution We have $T_{r+1} = {}^6C_r \left(\frac{3}{2}x^2\right)^{6-r} \left(-\frac{1}{3x}\right)^r$

$$= {}^6C_r \left(\frac{3}{2}\right)^{6-r} \left(x^2\right)^{6-r} (-1)^r \left(\frac{1}{x}\right)^r \left(\frac{1}{3^r}\right)$$

$$= (-1)^r \ {}^6C_r \ \frac{(3)^{6-2r}}{(2)^{6-r}} \ x^{12-3r}$$

The term will be independent of x if the index of x is zero, i.e., $12 - 3r = 0$. Thus, $r = 4$

Hence 5^{th} term is independent of x and is given by $(-1)^4 \ {}^6C_4 \ \frac{(3)^{6-8}}{(2)^{6-4}} = \frac{5}{12}$.

Example 11 If the coefficients of a^{r-1}, a^r and a^{r+1} in the expansion of $(1 + a)^n$ are in arithmetic progression, prove that $n^2 - n(4r + 1) + 4r^2 - 2 = 0$.

Solution The $(r + 1)^{th}$ term in the expansion is ${}^nC_r a^r$. Thus it can be seen that a^r occurs in the $(r + 1)^{th}$ term, and its coefficient is ${}^nC_r$. Hence the coefficients of a^{r-1}, a^r and a^{r+1} are ${}^nC_{r-1}$, ${}^nC_r$ and ${}^nC_{r+1}$, respectively. Since these coefficients are in arithmetic progression, so we have, ${}^nC_{r-1} + {}^nC_{r+1} = 2.{}^nC_r$. This gives

$$\frac{n!}{(r-1)!(n-r+1)!} + \frac{n!}{(r+1)!(n-r-1)!} = 2 \times \frac{n!}{r!(n-r)!}$$

i.e. $$\frac{1}{(r-1)!(n-r+1)(n-r)(n-r-1)!} + \frac{1}{(r+1)(r)(r-1)!(n-r-1)!}$$

$$= 2 \times \frac{1}{r(r-1)!(n-r)(n-r-1)!}$$

or $$\frac{1}{(r-1)! \ (n-r-1)!} \left[\frac{1}{(n-r)(n-r+1)} + \frac{1}{(r+1) \ (r)} \right]$$

$$= 2 \times \frac{1}{(r-1)! \ (n-r-1)![r(n-r)]}$$

i.e. $$\frac{1}{(n-r+1)(n-r)} + \frac{1}{r(r+1)} = \frac{2}{r(n-r)},$$

or $$\frac{r(r+1) + (n-r)(n-r+1)}{(n-r)(n-r+1)r(r+1)} = \frac{2}{r(n-r)}$$

or $$r(r + 1) + (n - r)(n - r + 1) = 2(r + 1)(n - r + 1)$$

or $$r^2 + r + n^2 - nr + n - nr + r^2 - r = 2(nr - r^2 + r + n - r + 1)$$

or $\quad n^2 - 4nr - n + 4r^2 - 2 = 0$

i.e., $\quad n^2 - n(4r + 1) + 4r^2 - 2 = 0$

Example 12 Show that the coefficient of the middle term in the expansion of $(1 + x)^{2n}$ is equal to the sum of the coefficients of two middle terms in the expansion of $(1 + x)^{2n-1}$.

Solution As $2n$ is even so the expansion $(1 + x)^{2n}$ has only one middle term which is

$\left(\frac{2n}{2}+1\right)^{\text{th}}$ i.e., $(n + 1)^{\text{th}}$ term.

The $(n + 1)^{\text{th}}$ term is ${}^{2n}C_n x^n$. The coefficient of x^n is ${}^{2n}C_n$

Similarly, $(2n - 1)$ being odd, the other expansion has two middle terms,

$\left(\frac{2n-1+1}{2}\right)^{\text{th}}$ and $\left(\frac{2n-1+1}{2}+1\right)^{\text{th}}$ i.e., n^{th} and $(n + 1)^{\text{th}}$ terms. The coefficients of these terms are ${}^{2n-1}C_{n-1}$ and ${}^{2n-1}C_n$, respectively.

Now

$${}^{2n-1}C_{n-1} + {}^{2n-1}C_n = {}^{2n}C_n \qquad [\text{As } {}^{n}C_{r-1} + {}^{n}C_r = {}^{n+1}C_r]. \text{ as required.}$$

Example 13 Find the coefficient of a^4 in the product $(1 + 2a)^4 (2 - a)^5$ using binomial theorem.

Solution We first expand each of the factors of the given product using Binomial Theorem. We have

$$(1 + 2a)^4 = {}^4C_0 + {}^4C_1 (2a) + {}^4C_2 (2a)^2 + {}^4C_3 (2a)^3 + {}^4C_4 (2a)^4$$

$$= 1 + 4(2a) + 6(4a^2) + 4(8a^3) + 16a^4.$$

$$= 1 + 8a + 24a^2 + 32a^3 + 16a^4$$

and $\quad (2 - a)^5 = {}^5C_0 (2)^5 - {}^5C_1 (2)^4 (a) + {}^5C_2 (2)^3 (a)^2 - {}^5C_3 (2)^2 (a)^3 + {}^5C_4 (2)(a)^4 - {}^5C_5 (a)^5$

$$= 32 - 80a + 80a^2 - 40a^3 + 10a^4 - a^5$$

Thus $(1 + 2a)^4 (2 - a)^5$

$$= (1 + 8a + 24a^2 + 32a^3 + 16a^4)(32 - 80a + 80a^2 - 40a^3 + 10a^4 - a^5)$$

The complete multiplication of the two brackets need not be carried out. We write only those terms which involve a^4. This can be done if we note that $a^r . a^{4-r} = a^4$. The terms containing a^4 are

$1(10a^4) + (8a)(-40a^3) + (24a^2)(80a^2) + (32a^3)(-80a) + (16a^4)(32) = -438a^4$

Thus, the coefficient of a^4 in the given product is – 438.

Example 14 Find the r^{th} term from the end in the expansion of $(x + a)^n$.

Solution There are $(n + 1)$ terms in the expansion of $(x + a)^n$. Observing the terms we can say that the first term from the end is the last term, i.e., $(n + 1)^{th}$ term of the expansion and $n + 1 = (n + 1) – (1 – 1)$. The second term from the end is the n^{th} term of the expansion, and $n = (n + 1) – (2 – 1)$. The third term from the end is the $(n – 1)^{th}$ term of the expansion and $n – 1 = (n + 1) – (3 – 1)$ and so on. Thus r^{th} term from the end will be term number $(n + 1) – (r – 1) = (n – r + 2)$ of the expansion. And the $(n – r + 2)^{th}$ term is ${}^nC_{n-r+1}\, x^{r-1}\, a^{n-r+1}$.

Example 15 Find the term independent of x in the expansion of $\left(\sqrt[3]{x}+\frac{1}{2\sqrt[3]{x}}\right)^{18}, x>0$.

Solution We have $T_{r+1} = {}^{18}C_r \left(\sqrt[3]{x}\right)^{18-r}\left(\frac{1}{2\sqrt[3]{x}}\right)^r$

$$= {}^{18}C_r\, x^{\frac{18-r}{3}} \cdot \frac{1}{2^r . x^{\frac{r}{3}}} = {}^{18}C_r \frac{1}{2^r} . x^{\frac{18-2r}{3}}$$

Since we have to find a term independent of x, i.e., term not having x, so take $\frac{18-2r}{3}=0$.

We get $r = 9$. The required term is ${}^{18}C_9 \frac{1}{2^9}$.

Example 16 The sum of the coefficients of the first three terms in the expansion of $\left(x-\frac{3}{x^2}\right)^m$, $x \neq 0$, m being a natural number, is 559. Find the term of the expansion containing x^3.

Solution The coefficients of the first three terms of $\left(x-\frac{3}{x^2}\right)^m$ are ${}^mC_0$, $(–3)\, {}^mC_1$ and $9\, {}^mC_2$. Therefore, by the given condition, we have

$${}^mC_0 - 3\, {}^mC_1 + 9\, {}^mC_2 = 559, \text{ i.e., } 1 - 3m + \frac{9m(m-1)}{2} = 559$$

which gives $m = 12$ (m being a natural number).

Now $$T_{r+1} = {}^{12}C_r\, x^{12-r}\left(-\frac{3}{x^2}\right)^r = {}^{12}C_r\,(-3)^r \cdot x^{12-3r}$$

Since we need the term containing x^3, so put $12 - 3r = 3$ i.e., $r = 3$.

Thus, the required term is ${}^{12}C_3\,(-3)^3\, x^3$, i.e., $-5940\, x^3$.

Example 17 If the coefficients of $(r-5)^{th}$ and $(2r-1)^{th}$ terms in the expansion of $(1 + x)^{34}$ are equal, find r.

Solution The coefficients of $(r-5)^{th}$ and $(2r-1)^{th}$ terms of the expansion $(1 + x)^{34}$ are ${}^{34}C_{r-6}$ and ${}^{34}C_{2r-2}$, respectively. Since they are equal so ${}^{34}C_{r-6} = {}^{34}C_{2r-2}$

Therefore, either $r - 6 = 2r - 2$ or $r-6 = 34 - (2r - 2)$

[Using the fact that if ${}^nC_r = {}^nC_p$, then either $r = p$ or $r = n - p$]

So, we get $r = -4$ or $r = 14$. r being a natural number, $r = -4$ is not possible. So, $r = 14$.

Miscellaneous Exercise on Chapter 8

1. Find a, b and n in the expansion of $(a + b)^n$ if the first three terms of the expansion are 729, 7290 and 30375, respectively.
2. Find a if the coefficients of x^2 and x^3 in the expansion of $(3 + ax)^9$ are equal.
3. Find the coefficient of x^5 in the product $(1 + 2x)^6\,(1 - x)^7$ using binomial theorem.
4. If a and b are distinct integers, prove that $a - b$ is a factor of $a^n - b^n$, whenever n is a positive integer.

   [**Hint** write $a^n = (a - b + b)^n$ and expand]
5. Evaluate $\left(\sqrt{3}+\sqrt{2}\right)^6 - \left(\sqrt{3}-\sqrt{2}\right)^6$.
6. Find the value of $\left(a^2+\sqrt{a^2-1}\right)^4 + \left(a^2-\sqrt{a^2-1}\right)^4$.
7. Find an approximation of $(0.99)^5$ using the first three terms of its expansion.
8. Find n, if the ratio of the fifth term from the beginning to the fifth term from the end in the expansion of $\left(\sqrt[4]{2}+\frac{1}{\sqrt[4]{3}}\right)^n$ is $\sqrt{6}:1$.