$$\tau = I\alpha \tag{7.43}$$

Eq. (7.43) is similar to Newton's second law for linear motion expressed symbolically as

$F = ma$

Just as force produces acceleration, torque produces angular acceleration in a body. The angular acceleration is directly proportional to the applied torque and is inversely proportional to the moment of inertia of the body. In this respect, Eq.(7.43) can be called Newton's second law for rotational motion about a fixed axis.

Example 7.15 A cord of negligible mass is wound round the rim of a fly wheel of mass 20 kg and radius 20 cm. A steady pull of 25 N is applied on the cord as shown in Fig. 7.35. The flywheel is mounted on a horizontal axle with frictionless bearings.

(a) Compute the angular acceleration of the wheel.

(b) Find the work done by the pull, when 2m of the cord is unwound.

(c) Find also the kinetic energy of the wheel at this point. Assume that the wheel starts from rest.

(d) Compare answers to parts (b) and (c).

Answer

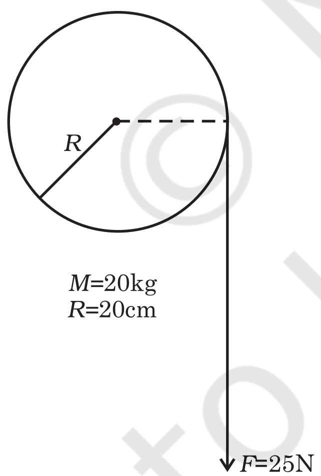

Fig. 7.35

(a) We use $I\alpha = \tau$

the torque $\tau = F R$

$= 25 \times 0.20$ Nm (as $R = 0.20$m)

$= 5.0$ Nm

I = Moment of inertia of flywheel about its axis $= \dfrac{MR^2}{2}$

$= \dfrac{20.0 \times (0.2)^2}{2} = 0.4$ kg m^2

α = angular acceleration

$= 5.0$ N m/0.4 kg m^2 = 12.5 s^{-2}

(b) Work done by the pull unwinding 2m of the cord

= 25 N × 2m = 50 J

(c) Let ω be the final angular velocity. The kinetic energy gained = $\dfrac{1}{2}I\omega^2$, since the wheel starts from rest. Now,

$\omega^2 = \omega_0^2 + 2\alpha\theta, \quad \omega_0 = 0$

The angular displacement θ = length of unwound string / radius of wheel

= 2m/0.2 m = 10 rad

$\omega^2 = 2 \times 12.5 \times 10.0 = 250 (\text{rad/s})^2$

$\therefore$ K.E. gained $= \dfrac{1}{2} \times 0.4 \times 250 = 50$ J

(d) The answers are the same, i.e. the kinetic energy gained by the wheel = work done by the force. There is no loss of energy due to friction. ◀

7.13 ANGULAR MOMENTUM IN CASE OF ROTATION ABOUT A FIXED AXIS

We have studied in section 7.7, the angular momentum of a system of particles. We already know from there that the time rate of total angular momentum of a system of particles about a point is equal to the total external torque on the system taken about the same point. When the total external torque is zero, the total angular momentum of the system is conserved.

We now wish to study the angular momentum in the special case of rotation about a fixed axis. The general expression for the total angular momentum of the system of n particles is

$$\mathbf{L} = \sum_{i=1}^{N} \mathbf{r}_i \times \mathbf{p}_i \tag{7.25b}$$

We first consider the angular momentum of a typical particle of the rotating rigid body. We then sum up the contributions of individual particles to get $\mathbf{L}$ of the whole body.

For a typical particle $\boldsymbol{l} = \mathbf{r} \times \mathbf{p}$. As seen in the last section $\mathbf{r} = \mathbf{OP} = \mathbf{OC} + \mathbf{CP}$ [Fig. 7.17(b)]. With

$\mathbf{p} = m\,\mathbf{v}$,

$\boldsymbol{l} = (\mathbf{OC} \times m\,\mathbf{v}) + (\mathbf{CP} \times m\,\mathbf{v})$

The magnitude of the linear velocity $\mathbf{v}$ of the particle at P is given by $v = \omega r_\perp$ where $r_\perp$ is the length of CP or the perpendicular distance of P from the axis of rotation. Further, $\mathbf{v}$ is tangential at P to the circle which the particle describes. Using the right-hand rule one can check that $\mathbf{CP} \times \mathbf{v}$ is parallel to the fixed axis. The unit vector along the fixed axis (chosen as the z-axis) is $\hat{\mathbf{k}}$. Hence

$$\mathbf{CP} \times m\,\mathbf{v} = r_\perp (mv)\,\hat{\mathbf{k}}$$

$$= m r_\perp^2 \omega\,\hat{\mathbf{k}} \quad (\text{since } v = \omega r_\perp)$$

Similarly, we can check that $\mathbf{OC} \times \mathbf{v}$ is perpendicular to the fixed axis. Let us denote the part of $\boldsymbol{l}$ along the fixed axis (i.e. the z-axis) by $\boldsymbol{l}_z$, then

$$\boldsymbol{l}_z = \mathbf{CP} \times m\,\mathbf{v} = m r_\perp^2 \omega\,\hat{\mathbf{k}}$$

and $\boldsymbol{l} = \boldsymbol{l}_z + \mathbf{OC} \times m\,\mathbf{v}$

We note that $\boldsymbol{l}_z$ is parallel to the fixed axis, but $\boldsymbol{l}$ is not. In general, for a particle, the angular momentum $\boldsymbol{l}$ is not along the axis of rotation, i.e. for a particle, $\boldsymbol{l}$ and $\boldsymbol{\omega}$ are not necessarily parallel. Compare this with the corresponding fact in translation. For a particle, $\mathbf{p}$ and $\mathbf{v}$ are always parallel to each other.

For computing the total angular momentum of the whole rigid body, we add up the contribution of each particle of the body.

Thus $\quad \mathbf{L} = \sum \boldsymbol{l}_i = \sum \boldsymbol{l}_{iz} + \sum \mathbf{OC}_i \times m_i \mathbf{v}_i$

We denote by $\mathbf{L}_\perp$ and $\mathbf{L}_z$ the components of $\mathbf{L}$ respectively perpendicular to the z-axis and along the z-axis;

$$\mathbf{L}_\perp = \sum \mathbf{OC}_i \times m_i \mathbf{v}_i \tag{7.44a}$$

where m_i and $\mathbf{v}_i$ are respectively the mass and the velocity of the i^{th} particle and C_i is the centre of the circle described by the particle;

and $$\mathbf{L}_z = \sum \boldsymbol{l}_{iz} = \left(\sum_i m_i r_i^2\right) w\,\mathbf{k}$$

or $$\mathbf{L}_z = I\omega\hat{\mathbf{k}} \tag{7.44b}$$

The last step follows since the perpendicular distance of the i^{th} particle from the axis is r_i; and by definition the moment of inertia of the body about the axis of rotation is $I = \sum m_i r_i^2$.

Note $\quad \mathbf{L} = \mathbf{L}_z + \mathbf{L}_\perp \quad$ (7.44c)

The rigid bodies which we have mainly considered in this chapter are symmetric about the axis of rotation, i.e. the axis of rotation is one of their symmetry axes. For such bodies, for a given $\mathbf{OC}_i$, for every particle which has a velocity $\mathbf{v}_i$, there is another particle of velocity $-\mathbf{v}_i$ located diametrically opposite on the circle with centre C_i described by the particle. Together such pairs will contribute zero to $\mathbf{L}_\perp$ and as a result for symmetric bodies $\mathbf{L}_\perp$ is zero, and hence

$$\mathbf{L} = \mathbf{L}_z = I\omega\hat{\mathbf{k}} \tag{7.44d}$$

For bodies, which are not symmetric about the axis of rotation, $\mathbf{L}$ is not equal to $\mathbf{L}_z$ and hence $\mathbf{L}$ does not lie along the axis of rotation.

Referring to Table 7.1, can you tell in which cases $\mathbf{L} = \mathbf{L}_z$ will not apply?

Let us differentiate Eq. (7.44b). Since $\hat{\mathbf{k}}$ is a fixed (constant) vector, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{L}_z) = \left(\frac{\mathrm{d}}{\mathrm{d}t}(I\omega)\right)\hat{\mathbf{k}}$$

Now, Eq. (7.28b) states

$$\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} = \boldsymbol{\tau}$$

As we have seen in the last section, only those components of the external torques which are along the axis of rotation, need to be taken into account, when we discuss rotation about a fixed axis. This means we can take $\boldsymbol{\tau} = \tau\hat{\mathbf{k}}$. Since $\mathbf{L} = \mathbf{L}_z + \mathbf{L}_\perp$ and the direction of $\mathbf{L}_z$ (vector $\hat{\mathbf{k}}$) is fixed, it follows that for rotation about a fixed axis,

$$\frac{\mathrm{d}\mathbf{L}_z}{\mathrm{d}t} = \tau\hat{\mathbf{k}} \tag{7.45a}$$

and $$\frac{\mathrm{d}\mathbf{L}_\perp}{\mathrm{d}t} = 0 \tag{7.45b}$$

Thus, for rotation about a fixed axis, the component of angular momentum perpendicular to the fixed axis is constant. As $\mathbf{L}_z = I\omega\hat{\mathbf{k}}$, we get from Eq. (7.45a),

$$\frac{\mathrm{d}}{\mathrm{d}t}(I\omega) = \tau \tag{7.45c}$$

If the moment of inertia I does not change with time,

$$\frac{d}{dt}(I\omega) = I\frac{d\omega}{dt} = I\alpha$$

and we get from Eq. (7.45c),

$$\tau = I\alpha \qquad (7.43)$$

We have already derived this equation using the work - kinetic energy route.

7.13.1 Conservation of angular momentum

We are now in a position to revisit the principle of conservation of angular momentum in the context of rotation about a fixed axis. From Eq. (7.45c), if the external torque is zero,

$$L_z = I\omega = \text{constant} \qquad (7.46)$$

For symmetric bodies, from Eq. (7.44d), L_z may be replaced by L .(L and L_z are respectively the magnitudes of $\mathbf{L}$ and $\mathbf{L}_z$.)

This then is the required form, for fixed axis rotation, of Eq. (7.29a), which expresses the general law of conservation of angular momentum of a system of particles. Eq. (7.46) applies to many situations that we come across in daily life. You may do this experiment with your friend. Sit on a swivel chair (a chair with a seat, free to rotate about a pivot) with your arms folded and feet not resting on, i.e., away from, the ground. Ask your friend to rotate the chair rapidly. While the chair is rotating with considerable angular speed stretch your arms horizontally. What happens? Your angular speed is reduced. If you bring back your arms closer to your body, the angular speed increases again. This is a situation where the principle of conservation of angular momentum is applicable. If friction in the rotational mechanism is neglected, there is no external torque about the axis of rotation of the chair and hence $I\omega$ is constant. Stretching the arms increases I about the axis of rotation, resulting in decreasing the angular speed ω. Bringing the arms closer to the body has the opposite effect.

A circus acrobat and a diver take advantage of this principle. Also, skaters and classical, Indian or western, dancers performing a pirouette (a spinning about a tip–top) on the toes of one foot display 'mastery' over this principle. Can you explain?

7.14 ROLLING MOTION

One of the most common motions observed in daily life is the rolling motion. All wheels used in transportation have rolling motion. For specificness we shall begin with the case of a disc, but the result will apply to any rolling body rolling on a level surface. We shall assume that the disc rolls without slipping. This means that at any instant of time the bottom of the disc

Fig 7.36 (a) *A demonstration of conservation of angular momentum. A girl sits on a swivel chair and stretches her arms/ brings her arms closer to the body.*

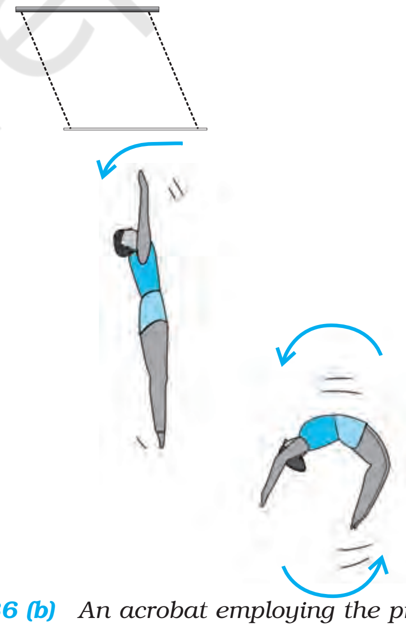

Fig 7.36 (b) *An acrobat employing the principle of conservation of angular momentum in her performance.*

which is in contact with the surface is at rest on the surface.

We have remarked earlier that rolling motion is a combination of rotation and translation. We know that the translational motion of a system of particles is the motion of its centre of mass.

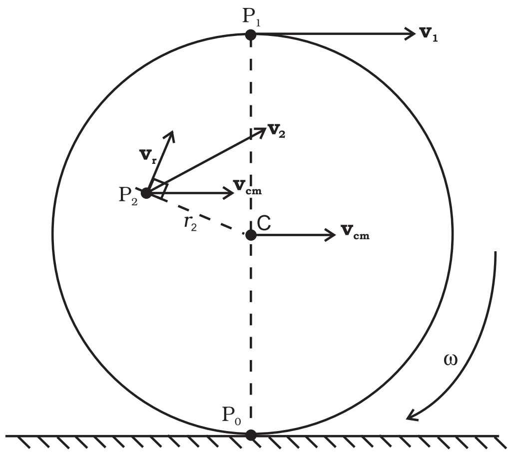

Fig. 7.37 *The rolling motion (without slipping) of a disc on a level surface. Note at any instant, the point of contact P_0 of the disc with the surface is at rest; the centre of mass of the disc moves with velocity, v_{cm}. The disc rotates with angular velocity ω about its axis which passes through C; v_{cm} =Rω, where R is the radius of the disc.*

Let $\mathbf{v}_{cm}$ be the velocity of the centre of mass and therefore the translational velocity of the disc. Since the centre of mass of the rolling disc is at its geometric centre C (Fig. 7. 37), $\mathbf{v}_{cm}$ is the velocity of C. It is parallel to the level surface. The rotational motion of the disc is about its symmetry axis, which passes through C. Thus, the velocity of any point of the disc, like P_0, P_1 or P_2, consists of two parts, one is the translational velocity $\mathbf{v}_{cm}$ and the other is the linear velocity $\mathbf{v}_r$ on account of rotation. The magnitude of $\mathbf{v}_r$ is $v_r = r\omega$, where ω is the angular velocity of the rotation of the disc about the axis and r is the distance of the point from the axis (i.e. from C). The velocity $\mathbf{v}_r$ is directed perpendicular to the radius vector of the given point with respect to C. In Fig. 7.37, the velocity of the point P_2 ($\mathbf{v}_2$) and its components $\mathbf{v}_r$ and $\mathbf{v}_{cm}$ are shown; $\mathbf{v}_r$ here is perpendicular to CP_2 . It is easy to show that $\mathbf{v}_z$ is perpendicular to the line P_OP_2. Therefore the line passing through P_O and parallel to $\boldsymbol{\omega}$ is called the instantaneous axis of rotation.

At P_o, the linear velocity, $\mathbf{v}_r$, due to rotation is directed exactly opposite to the translational velocity $\mathbf{v}_{cm}$. Further the magnitude of $\mathbf{v}_r$ here is $R\omega$, where R is the radius of the disc. The condition that P_o is instantaneously at rest requires $v_{cm} = R\omega$. Thus for the disc the condition for rolling without slipping is

$$v_{cm} = R\omega \qquad (7.47)$$

Incidentally, this means that the velocity of point P_1 at the top of the disc ($\mathbf{v}_1$) has a magnitude v_{cm}+ $R\omega$ or 2 v_{cm} and is directed parallel to the level surface. The condition (7.47) applies to all rolling bodies.

7.14.1 Kinetic Energy of Rolling Motion

Our next task will be to obtain an expression for the kinetic energy of a rolling body. The kinetic energy of a rolling body can be separated into kinetic energy of translation and kinetic energy of rotation. This is a special case of a general result for a system of particles, according to which the kinetic energy of a system of particles (K) can be separated into the kinetic energy of translational motion of the centre of mass ($MV^2/2$) and kinetic energy of rotational motion about the centre of mass of the system of particles (K'). Thus,

$$K = K' + MV^2/2 \qquad (7.48)$$

We assume this general result (see Exercise 7.31), and apply it to the case of rolling motion. In our notation, the kinetic energy of the centre of mass, i.e., the kinetic energy of translation, of the rolling body is mv^2_{cm} /2, where m is the mass of the body and v_{cm} is the centre of the mass velocity. Since the motion of the rolling body about the centre of mass is rotation, K' represents the kinetic energy of rotation of the body; $K' = I\omega^2/2$, where I is the moment of inertia about the appropriate axis, which is the symmetry axis of the rolling body. The kinetic energy of a rolling body, therefore, is given by

$$K = \frac{1}{2} I\omega^2 + \frac{1}{2} mv^2_{cm} \qquad (7.49a)$$

Substituting $I = mk^2$ where k = the corresponding radius of gyration of the body and $v_{cm} = R\,\omega$, we get

$$K = \frac{1}{2}\frac{mk^2 v^2_{cm}}{R^2} + \frac{1}{2} mv^2_{cm}$$

or $$K = \frac{1}{2} mv^2_{cm}\left(1 + \frac{k^2}{R^2}\right) \qquad (7.49b)$$

Equation (7.49b) applies to any rolling body: a disc, a cylinder, a ring or a sphere.

Example 7.16 Three bodies, a ring, a solid cylinder and a solid sphere roll down the same inclined plane without slipping. They start from rest. The radii of the bodies are identical. Which of the bodies reaches the ground with maximum velocity?

Answer We assume conservation of energy of the rolling body, i.e. there is no loss of energy due to friction etc. The potential energy lost by the body in rolling down the inclined plane (= mgh) must, therefore, be equal to kinetic energy gained. (See Fig.7.38) Since the bodies start from rest the kinetic energy gained is equal to the final kinetic energy of the bodies. From Eq. (7.49b), $K = \frac{1}{2}mv^2\left(1+\frac{k^2}{R^2}\right)$, where v is the final velocity of (the centre of mass of) the body. Equating K and mgh,

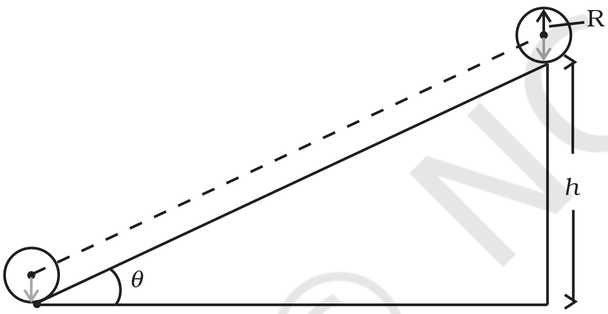

Fig.7.38

$$mgh = \frac{1}{2}mv^2\left(1+\frac{k^2}{R^2}\right)$$

$$\text{or } v^2 = \left(\frac{2gh}{1+k^2/R^2}\right)$$

Note is independent of the mass of the rolling body;

For a ring, $k^2 = R^2$

$$v_{ring} = \sqrt{\frac{2gh}{1+1}},$$

$$= \sqrt{gh}$$

For a solid cylinder $k^2 = R^2/2$

$$v_{disc} = \sqrt{\frac{2gh}{1+1/2}}$$

$$= \sqrt{\frac{4gh}{3}}$$

For a solid sphere $k^2 = 2R^2/5$

$$v_{sphere} = \sqrt{\frac{2gh}{1+2/5}}$$

$$= \sqrt{\frac{10gh}{7}}$$

From the results obtained it is clear that among the three bodies the sphere has the greatest and the ring has the least velocity of the centre of mass at the bottom of the inclined plane.

Suppose the bodies have the same mass. Which body has the greatest rotational kinetic energy while reaching the bottom of the inclined plane?

SUMMARY

1. Ideally, a rigid body is one for which the distances between different particles of the body do not change, even though there are forces on them.
2. A rigid body fixed at one point or along a line can have only rotational motion. A rigid body not fixed in some way can have either pure translational motion or a combination of translational and rotational motions.
3. In rotation about a fixed axis, every particle of the rigid body moves in a circle which lies in a plane perpendicular to the axis and has its centre on the axis. Every Point in the rotating rigid body has the same angular velocity at any instant of time.
4. In pure translation, every particle of the body moves with the same velocity at any instant of time.
5. Angular velocity is a vector. Its magnitude is $\omega = d\theta/dt$ and it is directed along the axis of rotation. For rotation about a fixed axis, this vector $\boldsymbol{\omega}$ has a fixed direction.