6.3 EXERCISE

SHORT ANSWER TYPE QUESTIONS

Q1. A spherical ball of salt is dissolving in water in such a manner that the rate of decrease of the volume at any instant is proportional to the surface. Prove that the radius is decreasing at a constant rate.

Sol. Ball of salt is spherical

$\therefore$ Volume of ball, $V = \frac{4}{3}\pi r^3$, where r = radius of the ball

As per the question, $\frac{dV}{dt} \propto S$, where S = surface area of the ball

$$\Rightarrow \quad \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right) \propto 4\pi r^2 \qquad [\because \quad S = 4\pi r^2]$$

$$\Rightarrow \quad \frac{4}{3}\pi \cdot 3r^2 \cdot \frac{dr}{dt} \propto 4\pi r^2$$

$$\Rightarrow \quad 4\pi r^2 \cdot \frac{dr}{dt} = K \cdot 4\pi r^2 \qquad (K = \text{Constant of proportionality})$$

$$\Rightarrow \quad \frac{dr}{dt} = K \cdot \frac{4\pi r^2}{4\pi r^2}$$

$$\therefore \quad \frac{dr}{dt} = K \cdot 1 = K$$

Hence, the radius of the ball is decreasing at constant rate.

Q2. If the area of a circle increases at a uniform rate, then prove that perimeter varies inversely as the radius.

Sol. We know that:

Area of circle, $A = \pi r^2$, where r = radius of the circle.

and perimeter $= 2\pi r$

As per the question,

$$\frac{dA}{dt} = K, \text{ where } K = \text{constant}$$

$$\Rightarrow \quad \frac{d}{dt}(\pi r^2) = K \quad \Rightarrow \quad \pi \cdot 2r \cdot \frac{dr}{dt} = K$$

$$\therefore \quad \frac{dr}{dt} = \frac{K}{2\pi r} \qquad ...(1)$$

Now Perimeter $\quad c = 2\pi r$

Differentiating both sides w.r.t., t, we get

$$\Rightarrow \quad \frac{dc}{dt} = \frac{d}{dt}(2\pi r) \quad \Rightarrow \quad \frac{dc}{dt} = 2\pi \cdot \frac{dr}{dt}$$

$$\Rightarrow \quad \frac{dc}{dt} = 2\pi \cdot \frac{K}{2\pi r} = \frac{K}{r} \qquad \text{[From (1)]}$$

$$\Rightarrow \quad \frac{dc}{dt} \propto \frac{1}{r}$$

Hence, the perimeter of the circle varies inversely as the radius of the circle.

Q3. A kite is moving horizontally at a height of 151.5 metres. If the speed of the kite is 10 m/s, how fast is the string being let out; when the kite is 250 m away from the boy who is flying the kite? The height of the boy is 1.5 m.

Sol. Given that height of the kite (h) = 151.5 m

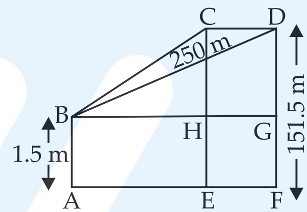
Speed of the kite(V) = 10 m/s

Let FD be the height of the kite and AB be the height of the boy.

Let AF = x m

$\therefore$ BG = AF = x m

and $\frac{dx}{dt} = 10$ m/s



From the figure, we get that

$$GD = DF - GF \Rightarrow DF - AB$$

$$= (151.5 - 1.5) \text{ m} = 150 \text{ m} \qquad [\because \quad AB = GF]$$

Now in ΔBGD,

$$BG^2 + GD^2 = BD^2 \qquad \text{(By Pythagoras Theorem)}$$

$$\Rightarrow \quad x^2 + (150)^2 = (250)^2$$

$$\Rightarrow \quad x^2 + 22500 = 62500 \quad \Rightarrow x^2 = 62500 - 22500$$

$$\Rightarrow \quad x^2 = 40000 \quad \Rightarrow x = 200 \text{ m}$$

Let initially the length of the string be y m

$\therefore$ In ΔBGD

$$BG^2 + GD^2 = BD^2 \quad \Rightarrow x^2 + (150)^2 = y^2$$

Differentiating both sides w.r.t., t, we get

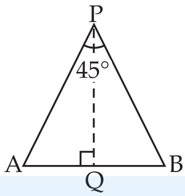
$$\Rightarrow \quad 2x \cdot \frac{dx}{dt} + 0 = 2y \cdot \frac{dy}{dt} \qquad \left[\because \frac{dx}{dt} = 10 \text{ m/s}\right]$$

$$\Rightarrow \quad 2 \times 200 \times 10 = 2 \times 250 \times \frac{dy}{dt}$$

$$\therefore \quad \frac{dy}{dt} = \frac{2 \times 200 \times 10}{2 \times 250} = 8 \text{ m/s}$$

Hence, the rate of change of the length of the string is 8 m/s.

Q4. Two men A and B start with velocities V at the same time from the junction of two roads inclined at 45° to each other. If they travel by different roads, find the rate at which they are being separated.



Sol. Let P be any point at which the two roads are inclined at an angle of 45°.

Two men A and B are moving along the roads PA and PB respectively with the same speed 'V'.

Let A and B be their final positions such that

$AB = y$

$\angle APB = 45°$ and they move with the same speed.

$\therefore$ ΔAPB is an isosceles triangle. Draw $PQ \perp AB$

$AB = y \quad \therefore \quad AQ = \frac{y}{2}$ and $PA = PB = x$ (let)

$$\angle APQ = \angle BPQ = \frac{45}{2} = 22\frac{1}{2}^\circ$$

[$\because$ In an isosceles Δ, the altitude drawn from the vertex, bisects the base]

Now in right ΔAPQ,

$$\sin 22\frac{1}{2}^\circ = \frac{AQ}{AP}$$

$$\Rightarrow \quad \sin 22\frac{1}{2}^\circ = \frac{\frac{y}{2}}{x} = \frac{y}{2x} \quad \Rightarrow y = 2x \cdot \sin 22\frac{1}{2}^\circ$$

Differentiating both sides w.r.t, t, we get

$$\frac{dy}{dt} = 2 \cdot \frac{dx}{dt} \cdot \sin 22\frac{1}{2}^\circ$$

$$= 2 \cdot V \cdot \frac{\sqrt{2-\sqrt{2}}}{2} \quad \left[\because \sin 22\frac{1}{2}^\circ = \frac{\sqrt{2-\sqrt{2}}}{2}\right]$$

$$= \sqrt{2-\sqrt{2}}\ V \text{ m/s}$$

Hence, the rate of their separation is $\sqrt{2-\sqrt{2}}$ V unit/s.

Q5. Find an angle θ, $0 < \theta < \frac{\pi}{2}$, which increases twice as fast as its sine.

Sol. As per the given condition,

$$\frac{d\theta}{dt} = 2\frac{d}{dt}(\sin\theta)$$

$$\Rightarrow \quad \frac{d\theta}{dt} = 2\cos\theta \cdot \frac{d\theta}{dt} \quad \Rightarrow 1 = 2\cos\theta$$

$$\therefore \quad \cos\theta = \frac{1}{2} \quad \Rightarrow \quad \cos\theta = \cos\frac{\pi}{3} \quad \Rightarrow \theta = \frac{\pi}{3}$$

Hence, the required angle is $\frac{\pi}{3}$.

Q6. Find the approximate value of $(1.999)^5$.

Sol. $(1.999)^5 = (2 - 0.001)^5$

Let $x = 2$ and $\Delta x = -0.001$

Let $y = x^5$

Differentiating both sides w.r.t, x, we get

$$\frac{dy}{dx} = 5x^4 = 5(2)^4 = 80$$

Now $\Delta y = \left(\frac{dy}{dx}\right) \cdot \Delta x = 80 \cdot (-0.001) = -0.080$

$\therefore \quad (1.999)^5 = y + \Delta y$

$= x^5 - 0.080 = (2)^5 - 0.080 = 32 - 0.080 = 31.92$

Hence, approximate value of $(1.999)^5$ is 31.92.

Q7. Find the approximate volume of metal in a hollow spherical shell whose internal and external radii are 3 cm and 3.0005 cm respectively.

Sol. Internal radius $r = 3$ cm

and external radius $R = r + \Delta r = 3.0005$ cm

$\therefore \quad \Delta r = 3.0005 - 3 = 0.0005$ cm

Let $y = r^3 \Rightarrow y + \Delta y = (r + \Delta r)^3 = R^3 = (3.0005)^3 \quad ...(i)$

Differentiating both sides w.r.t., r, we get

$$\frac{dy}{dr} = 3r^2$$

$\therefore \quad \Delta y = \frac{dy}{dr} \times \Delta r = 3r^2 \times 0.0005$

$= 3 \times (3)^2 \times 0.0005 = 27 \times 0.0005 = 0.0135$

$\therefore \quad (3.0005)^3 = y + \Delta y$ [From eq. (i)]

$= (3)^3 + 0.0135 = 27 + 0.0135 = 27.0135$

Volume of the shell $= \frac{4}{3}\pi[R^3 - r^3]$

$= \frac{4}{3}\pi\,[27.0135 - 27] = \frac{4}{3}\pi \times 0.0135$

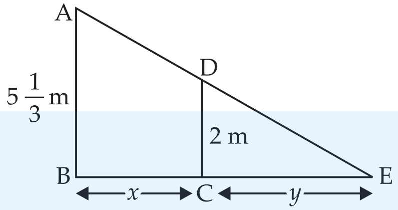
$= 4\pi \times 0.005 = 4 \times 3.14 \times 0.0045 = 0.018\,\pi\,\text{cm}^3$

Hence, the approximate volume of the metal in the shell is 0.018π cm^3.

Q8. A man, 2m tall, walks at the rate of $1\frac{2}{3}$ m/s towards a street light which is $5\frac{1}{3}$ m above the ground. At what rate is the tip of his shadow moving? At what rate is the length of the shadow changing when he is $3\frac{1}{3}$ m from the base of the light?

Sol. Let AB is the height of street light post and CD is the height of the man such that

$AB = 5\frac{1}{3} = \frac{16}{3}$ m and CD = 2 m



Let BC = x length (the distance of the man from the lamp post) and CE = y is the length of the shadow of the man at any instant.

From the figure, we see that

$\Delta ABE \sim \Delta DCE$ [by AAA Similarity]

∴ Taking ratio of their corresponding sides, we get

$$\frac{AB}{CD} = \frac{BE}{CE} \Rightarrow \frac{AB}{CD} = \frac{BC + CE}{CE}$$

$$\Rightarrow \frac{16/3}{2} = \frac{x+y}{y} \Rightarrow \frac{8}{3} = \frac{x+y}{y}$$

$$\Rightarrow 8y = 3x + 3y \Rightarrow 8y - 3y = 3x \Rightarrow 5y = 3x$$

Differentiating both sides w.r.t, t, we get

$$\cdot \frac{dy}{dt} = 3 \cdot \frac{dx}{dt}$$

$$\Rightarrow \frac{dy}{dt} = \frac{3}{5} \cdot \frac{dx}{dt} \Rightarrow \frac{dy}{dt} = \frac{3}{5} \cdot \left(-1\frac{2}{3}\right) = \frac{3}{5} \cdot \left(\frac{-5}{3}\right)$$

[∵ man is moving in opposite direction]

$= -1$ m/s

Hence, the length of shadow is decreasing at the rate of 1 m/s.

Now let $u = x + y$

(u = distance of the tip of shadow from the light post)

Differentiating both sides w.r.t. t, we get

$$\frac{du}{dt} = \frac{dx}{dt} + \frac{dy}{dt}$$

$$= \left(-1\frac{2}{3} - 1\right) = -\left(\frac{5}{3} + 1\right) = -\frac{8}{3} = -2\frac{2}{3} \text{ m/s}$$

Hence, the tip of the shadow is moving at the rate of $2\frac{2}{3}$ m/s towards the light post and the length of shadow decreasing at the rate of 1 m/s.

Q9. A swimming pool is to be drained for cleaning. If L represents the number of litres of water in the pool t seconds after the pool has been plugged off to drain and $L = 200(10 - t)^2$. How fast is the water running out at the end of 5 seconds? What is the average rate at which the water flows out during the first 5 seconds?

Sol. Given that $L = 200(10 - t)^2$

where L represents the number of litres of water in the pool.

Differentiating both sides w.r.t, t, we get

$$\frac{dL}{dt} = 200 \times 2(10 - t)(-1) = -400(10 - t)$$

But the rate at which the water is running out

$$= -\frac{dL}{dt} = 400(10 - t) \quad \text{...(1)}$$

Rate at which the water is running after 5 seconds

$$= 400 \times (10 - 5) = 2000 \text{ L/s (final rate)}$$

For initial rate put $t = 0$

$$= 400(10 - 0) = 4000 \text{ L/s}$$

The average rate at which the water is running out

$$= \frac{\text{Initial rate} + \text{Final rate}}{2} = \frac{4000 + 2000}{2} = \frac{6000}{2} = 3000 \text{ L/s}$$

Hence, the required rate = 3000 L/s.

Q10. The volume of a cube increases at a constant rate. Prove that the increase in its surface area varies inversely as the length of the side.

Sol. Let x be the length of the cube

$\therefore$ Volume of the cube $V = x^3$ ...(1)

Given that $\frac{dV}{dt} = K$

Differentiating Eq. (1) w.r.t. t, we get

$$\frac{dV}{dt} = 3x^2 \cdot \frac{dx}{dt} = K \text{ (constant)}$$

$$\therefore \quad \frac{dx}{dt} = \frac{K}{3x^2}$$

Now surface area of the cube, $S = 6x^2$

Differentiating both sides w.r.t. t, we get

$$\frac{ds}{dt} = 6 \cdot 2 \cdot x \cdot \frac{dx}{dt} = 12x \cdot \frac{K}{3x^2}$$

$$\Rightarrow \quad \frac{ds}{dt} = \frac{4K}{x} \Rightarrow \frac{ds}{dt} \propto \frac{1}{x} \quad (4K = \text{constant})$$

Hence, the surface area of the cube varies inversely as the length of the side.

Q11. x and y are the sides of two squares such that $y = x - x^2$. Find the rate of change of the area of second square with respect to the area of first square.

Sol. Let area of the first square $A_1 = x^2$
and area of the second square $A_2 = y^2$
Now $A_1 = x^2$ and $A_2 = y^2 = (x - x^2)^2$
Differentiating both A_1 and A_2 w.r.t. t, we get

$$\frac{dA_1}{dt} = 2x \cdot \frac{dx}{dt} \text{ and } \frac{dA_2}{dt} = 2(x - x^2)(1 - 2x) \cdot \frac{dx}{dt}$$

$$\therefore \quad \frac{dA_2}{dA_1} = \frac{\frac{dA_2}{dt}}{\frac{dA_1}{dt}} = \frac{2(x - x^2)(1 - 2x) \cdot \frac{dx}{dt}}{2x \cdot \frac{dx}{dt}}$$

$$= \frac{x(1 - x)(1 - 2x)}{x} = (1 - x)(1 - 2x)$$

$$= 1 - 2x - x + 2x^2 = 2x^2 - 3x + 1$$

Hence, the rate of change of area of the second square with respect to first is $2x^2 - 3x + 1$.

Q12. Find the condition that the curves $2x = y^2$ and $2xy = k$ intersect orthogonally.

Sol. The two circles intersect orthogonally if the angle between the tangents drawn to the two circles at the point of their intersection is 90°.
Equation of the two circles are given as

$$2x = y^2 \quad \text{...(i)}$$

and $$2xy = k \quad \text{...}(ii)$$

Differentiating eq. (i) and (ii) w.r.t. x, we get

$$2.1 = 2y \cdot \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{y} \Rightarrow m_1 = \frac{1}{y}$$

(m_1 = slope of the tangent)

$$\Rightarrow \quad 2xy = k$$

$$\Rightarrow \quad 2\left[x \cdot \frac{dy}{dx} + y \cdot 1\right] = 0$$

$$\therefore \quad \frac{dy}{dx} = -\frac{y}{x} \Rightarrow m_2 = -\frac{y}{x}$$

[m_2 = slope of the other tangent]

If the two tangents are perpendicular to each other,
then $m_1 \times m_2 = -1$

$$\Rightarrow \quad \frac{1}{y} \times \left(-\frac{y}{x}\right) = -1 \Rightarrow \frac{1}{x} = 1 \Rightarrow x = 1$$

Now solving $2x = y^2$ [From (*i*)]

and $2xy = k$ [From (*ii*)]

From eq. (*ii*) $y = \frac{k}{2x}$

Putting the value of y in eq. (*i*)

$$2x = \left(\frac{k}{2x}\right)^2 \Rightarrow 2x = \frac{k^2}{4x^2}$$

$\Rightarrow \quad 8x^3 = k^2 \Rightarrow 8(1)^3 = k^2 \Rightarrow 8 = k^2$

Hence, the required condition is $k^2 = 8$.

Q13. Prove that the curves $xy = 4$ and $x^2 + y^2 = 8$ touch each other.

Sol. Given circles are $xy = 4$ ...(*i*)

and $x^2 + y^2 = 8$ ...(*ii*)

Differentiating eq. (*i*) w.r.t., x

$$x \cdot \frac{dy}{dx} + y \cdot 1 = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{y}{x} \Rightarrow m_1 = -\frac{y}{x} \qquad ...(iii)$$

where, m_1 is the slope of the tangent to the curve.

Differentiating eq. (*ii*) w.r.t. x

$$2x + 2y \cdot \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{x}{y} \Rightarrow m_2 = -\frac{x}{y}$$

where, m_2 is the slope of the tangent to the circle.

To find the point of contact of the two circles

$$m_1 = m_2 \Rightarrow -\frac{y}{x} = -\frac{x}{y} \Rightarrow x^2 = y^2$$

Putting the value of y^2 in eq. (*ii*)

$$x^2 + x^2 = 8 \Rightarrow 2x^2 = 8 \Rightarrow x^2 = 4$$

$\therefore \quad x = \pm 2$

$\because \quad x^2 = y^2 \Rightarrow y = \pm 2$

$\therefore$ The point of contact of the two circles are (2, 2) and (−2, 2).

Q14. Find the coordinates of the point on the curve $\sqrt{x} + \sqrt{y} = 4$ at which tangent is equally inclined to the axes.

Sol. Equation of curve is given by $\sqrt{x} + \sqrt{y} = 4$

Let (x_1, y_1) be the required point on the curve

$\therefore \quad \sqrt{x_1} + \sqrt{y_1} = 4$

Differentiating both sides w.r.t. x_1, we get

$$\frac{d}{dx_1}\sqrt{x_1} + \frac{d}{dx_1}\sqrt{y_1} = \frac{d}{dx_1}(4)$$

$\Rightarrow \quad \frac{1}{2\sqrt{x_1}} + \frac{1}{2\sqrt{y_1}} \cdot \frac{dy_1}{dx_1} = 0$

$\Rightarrow \quad \frac{1}{\sqrt{x_1}} + \frac{1}{\sqrt{y_1}} \cdot \frac{dy_1}{dx_1} = 0 \quad \Rightarrow \quad \frac{dy_1}{dx_1} = -\frac{\sqrt{y_1}}{\sqrt{x_1}}$ ...(*i*)

Since the tangent to the given curve at (x_1, y_1) is equally inclined to the axes.

$\therefore$ Slope of the tangent $\frac{dy_1}{dx_1} = \pm \tan\frac{\pi}{4} = \pm 1$

So, from eq. (*i*) we get

$$-\frac{\sqrt{y_1}}{\sqrt{x_1}} = \pm 1$$

Squaring both sides, we get

$$\frac{y_1}{x_1} = 1 \quad \Rightarrow \quad y_1 = x_1$$

Putting the value of y_1 in the given equation of the curve.

$\sqrt{x_1} + \sqrt{y_1} = 4$

$\Rightarrow \quad \sqrt{x_1} + \sqrt{x_1} = 4 \quad \Rightarrow \quad 2\sqrt{x_1} = 4 \quad \Rightarrow \quad \sqrt{x_1} = 2 \quad \Rightarrow \quad x_1 = 4$

Since $\quad y_1 = x_1$

$\therefore \quad y_1 = 4$

Hence, the required point is (4, 4).

Q15. Find the angle of intersection of the curves $y = 4 - x^2$ and $y = x^2$.

Sol. We know that the angle of intersection of two curves is equal to the angle between the tangents drawn to the curves at their point of intersection.

The given curves are $y = 4 - x^2$ ... (*i*) and $y = x^2$ ...(*ii*)

Differentiating eq. (*i*) and (*ii*) with respect to x, we have

$$\frac{dy}{dx} = -2x \quad \Rightarrow \quad m_1 = -2x$$

m_1 is the slope of the tangent to the curve (*i*).

and $\quad \frac{dy}{dx} = 2x \quad \Rightarrow \quad m_2 = 2x$

m_2 is the slope of the tangent to the curve (*ii*).

So, $\quad m_1 = -2x$ and $m_2 = 2x$

Now solving eq. (*i*) and (*ii*) we get

$\Rightarrow \quad 4 - x^2 = x^2 \quad \Rightarrow \quad 2x^2 = 4 \quad \Rightarrow \quad x^2 = 2 \quad \Rightarrow \quad x = \pm\sqrt{2}$

So, $\quad m_1 = -2x = -2\sqrt{2}$ and $m_2 = 2x = 2\sqrt{2}$

Let θ be the angle of intersection of two curves

$\therefore \quad \tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right|$

$= \left|\frac{2\sqrt{2} + 2\sqrt{2}}{1 - (2\sqrt{2})(2\sqrt{2})}\right| = \left|\frac{4\sqrt{2}}{1-8}\right| = \left|\frac{4\sqrt{2}}{-7}\right| = \frac{4\sqrt{2}}{7}$

$\therefore \quad \theta = \tan^{-1}\left(\frac{4\sqrt{2}}{7}\right)$

Hence, the required angle is $\tan^{-1}\left(\frac{4\sqrt{2}}{7}\right)$.

Q16. Prove that the curves $y^2 = 4x$ and $x^2 + y^2 - 6x + 1 = 0$ touch each other at the point (1, 2).

Sol. Given that the equation of the two curves are $y^2 = 4x$ ...(*i*)

and $x^2 + y^2 - 6x + 1 = 0$ ...(*ii*)

Differentiating (*i*) w.r.t. x, we get $2y\frac{dy}{dx} = 4 \Rightarrow \frac{dy}{dx} = \frac{2}{y}$

Slope of the tangent at (1, 2), $m_1 = \frac{2}{2} = 1$

Differentiating (*ii*) w.r.t. $x \Rightarrow 2x + 2y\cdot\frac{dy}{dx} - 6 = 0$

$\Rightarrow \quad 2y\cdot\frac{dy}{dx} = 6 - 2x \quad \Rightarrow \quad \frac{dy}{dx} = \frac{6-2x}{2y}$

$\therefore$ Slope of the tangent at the same point (1, 2)

$\Rightarrow \quad m_2 = \frac{6 - 2\times 1}{2\times 2} = \frac{4}{4} = 1$

We see that $m_1 = m_2 = 1$ at the point (1, 2).

Hence, the given circles touch each other at the same point (1, 2).

Q17. Find the equation of the normal lines to the curve $3x^2 - y^2 = 8$ which are parallel to the line $x + 3y = 4$.

Sol. We have equation of the curve $3x^2 - y^2 = 8$

Differentiating both sides w.r.t. x, we get

$\Rightarrow \quad 6x - 2y\cdot\frac{dy}{dx} = 0 \quad \Rightarrow \quad -2y\frac{dy}{dx} = -6x \quad \Rightarrow \quad \frac{dy}{dx} = \frac{3x}{y}$

Slope of the tangent to the given curve $= \frac{3x}{y}$

$\therefore$ Slope of the normal to the curve $= -\frac{1}{\frac{3x}{y}} = -\frac{y}{3x}$.

Now differentiating both sides the given line $x + 3y = 4$

$\Rightarrow \quad 1 + 3 \cdot \frac{dy}{dx} = 0 \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{1}{3}$

Since the normal to the curve is parallel to the given line $x + 3y = 4$.

$\therefore \quad -\frac{y}{3x} = -\frac{1}{3} \quad \Rightarrow \quad y = x$

Putting the value of y in $3x^2 - y^2 = 8$, we get

$3x^2 - x^2 = 8 \quad \Rightarrow 2x^2 = 8 \quad \Rightarrow \quad x^2 = 4 \quad \Rightarrow \quad x = \pm 2$

$\therefore \quad y = \pm 2$

$\therefore$ The points on the curve are (2, 2) and (– 2, – 2).

Now equation of the normal to the curve at (2, 2) is

$y - 2 = -\frac{1}{3}(x - 2)$

$\Rightarrow \quad 3y - 6 = -x + 2 \quad \Rightarrow x + 3y = 8$

at (– 2, – 2) $\quad y + 2 = -\frac{1}{3}(x + 2)$

$\Rightarrow \quad 3y + 6 = -x - 2 \quad \Rightarrow x + 3y = -8$

Hence, the required equations are $x + 3y = 8$ and $x + 3y = -8$ or $x + 3y = \pm 8$.

Q18. At what points on the curve $x^2 + y^2 - 2x - 4y + 1 = 0$, the tangents are parallel to the y-axis?

Sol. Given that the equation of the curve is

$x^2 + y^2 - 2x - 4y + 1 = 0 \quad$ ...(*i*)

Differentiating both sides w.r.t. x, we have

$2x + 2y \cdot \frac{dy}{dx} - 2 - 4 \cdot \frac{dy}{dx} = 0$

$\Rightarrow \quad (2y - 4)\frac{dy}{dx} = 2 - 2x \Rightarrow \frac{dy}{dx} = \frac{2 - 2x}{2y - 4} \quad$ ...(*ii*)

Since the tangent to the curve is parallel to the y-axis.

$\therefore \quad \text{Slope } \frac{dy}{dx} = \tan\frac{\pi}{2} = \infty = \frac{1}{0}$

So, from eq. (*ii*) we get

$\frac{2 - 2x}{2y - 4} = \frac{1}{0} \quad \Rightarrow \quad 2y - 4 = 0 \quad \Rightarrow \quad y = 2$

Now putting the value of y in eq. (*i*), we get

$\Rightarrow \quad x^2 + (2)^2 - 2x - 8 + 1 = 0$

$\Rightarrow \quad x^2 - 2x + 4 - 8 + 1 = 0$

$\Rightarrow \quad x^2 - 2x - 3 = 0 \quad \Rightarrow \quad x^2 - 3x + x - 3 = 0$

$\Rightarrow \quad x(x - 3) + 1(x - 3) = 0 \quad \Rightarrow (x - 3)(x + 1) = 0$

$\Rightarrow \quad x = -1 \text{ or } 3$

Hence, the required points are (– 1, 2) and (3, 2).

Q19. Show that the line $\frac{x}{a}+\frac{y}{b}=1$, touches the curve $y = b \cdot e^{-x/a}$ at the point where the curve intersects the axis of y.

Sol. Given that $y = b \cdot e^{-x/a}$, the equation of curve

and $\frac{x}{a}+\frac{y}{b}=1$, the equation of line.

Let the coordinates of the point where the curve intersects the y-axis be $(0, y_1)$

Now differentiating $y = b \cdot e^{-x/a}$ both sides w.r.t. x, we get

$$\frac{dy}{dx} = b \cdot e^{-x/a}\left(-\frac{1}{a}\right) = -\frac{b}{a} \cdot e^{-x/a}$$

So, the slope of the tangent, $m_1 = -\frac{b}{a} e^{-x/a}$.

Differentiating $\frac{x}{a}+\frac{y}{b}=1$ both sides w.r.t. x, we get

$$\frac{1}{a}+\frac{1}{b}\cdot\frac{dy}{dx}=0$$

So, the slope of the line, $m_2 = \frac{-b}{a}$.

If the line touches the curve, then $m_1 = m_2$

$$\Rightarrow \quad \frac{-b}{a}\cdot e^{-x/a} = \frac{-b}{a} \quad \Rightarrow \quad e^{-x/a} = 1$$

$$\Rightarrow \quad \frac{-x}{a}\log e = \log 1 \qquad \text{(Taking log on both sides)}$$

$$\Rightarrow \quad \frac{-x}{a} = 0 \quad \Rightarrow \quad x = 0$$

Putting $x = 0$ in equation $y = b \cdot e^{-x/a}$

$$\Rightarrow \quad y = b \cdot e^0 = b$$

Hence, the given equation of curve intersect at $(0, b)$ i.e. on y-axis.

Q20. Show that $f(x) = 2x + \cot^{-1} x + \log(\sqrt{1+x^2} - x)$ is increasing in **R**.

Sol. Given that $f(x) = 2x + \cot^{-1} x + \log(\sqrt{1+x^2} - x)$

Differentiating both sides w.r.t. x, we get

$$f'(x) = 2 - \frac{1}{1+x^2} + \frac{1}{\sqrt{1+x^2}-x} \times \frac{d}{dx}\left(\sqrt{1+x^2}-x\right)$$

$$= 2 - \frac{1}{1+x^2} + \frac{\left(\frac{1}{2\sqrt{1+x^2}} \times (2x-1)\right)}{\sqrt{1+x^2}-x}$$

$$= 2 - \frac{1}{1+x^2} + \frac{x - \sqrt{1+x^2}}{\sqrt{1+x^2}\left(\sqrt{1+x^2} - x\right)}$$

$$= 2 - \frac{1}{1+x^2} - \frac{\left(\sqrt{1+x^2} - x\right)}{\sqrt{1+x^2}\left(\sqrt{1+x^2} - x\right)}$$

$$= 2 - \frac{1}{1+x^2} - \frac{1}{\sqrt{1+x^2}}$$

For increasing function, $f'(x) \geq 0$

$\therefore \quad 2 - \frac{1}{1+x^2} - \frac{1}{\sqrt{1+x^2}} \geq 0$

$\Rightarrow \quad \frac{2(1+x^2) - 1 + \sqrt{1+x^2}}{(1+x^2)} \geq 0 \quad \Rightarrow \quad 2 + 2x^2 - 1 + \sqrt{1+x^2} \geq 0$

$\Rightarrow \quad 2x^2 + 1 + \sqrt{1+x^2} \geq 0 \quad \Rightarrow \quad 2x^2 + 1 \geq -\sqrt{1+x^2}$

Squaring both sides, we get $4x^4 + 1 + 4x^2 \geq 1 + x^2$

$\Rightarrow \quad 4x^4 + 4x^2 - x^2 \geq 0 \quad \Rightarrow \quad 4x^4 + 3x^2 \geq 0 \quad \Rightarrow x^2(4x^2 + 3) \geq 0$

which is true for any value of $x \in$ R.

Hence, the given function is an increasing function over R.

Q21. Show that for $a \geq 1$, $f(x) = \sqrt{3}\sin x - \cos x - 2ax + b$ is decreasing in **R**.

Sol. Given that: $f(x) = \sqrt{3}\sin x - \cos x - 2ax + b$, $a \geq 1$

Differentiating both sides w.r.t. x, we get

$$f'(x) = \sqrt{3}\cos x + \sin x - 2a$$

For decreasing function, $f'(x) < 0$

$\therefore \quad \sqrt{3}\cos x + \sin x - 2a < 0$

$\Rightarrow \quad 2\left(\frac{\sqrt{3}}{2}\cos x + \frac{1}{2}\sin x\right) - 2a < 0$

$\Rightarrow \quad \frac{\sqrt{3}}{2}\cos x + \frac{1}{2}\sin x - a < 0$

$\Rightarrow \quad \left(\cos\frac{\pi}{6}\cos x + \sin\frac{\pi}{6}\sin x\right) - a < 0$

$\Rightarrow \quad \cos\left(x - \frac{\pi}{6}\right) - a < 0$

Since $\cos x \in [-1, 1]$ and $a \geq 1$

$\therefore \quad f'(x) < 0$

Hence, the given function is decreasing in R.

Q22. Show that $f(x) = \tan^{-1}(\sin x + \cos x)$ is an increasing function in $\left(0, \frac{\pi}{4}\right)$.

Sol. Given that: $f(x) = \tan^{-1}(\sin x + \cos x)$ in $\left(0, \frac{\pi}{4}\right)$

Differentiating both sides w.r.t. x, we get

$$f'(x) = \frac{1}{1+(\sin x+\cos x)^2} \cdot \frac{d}{dx}(\sin x + \cos x)$$

$$\Rightarrow \quad f'(x) = \frac{1\times(\cos x - \sin x)}{1+(\sin x+\cos x)^2}$$

$$\Rightarrow \quad f'(x) = \frac{\cos x - \sin x}{1+\sin^2 x+\cos^2 x + 2\sin x\cos x}$$

$$\Rightarrow \quad f'(x) = \frac{\cos x - \sin x}{1+1+2\sin x\cos x} \quad \Rightarrow f'(x) = \frac{\cos x - \sin x}{2+2\sin x\cos x}$$

For an increasing function $f'(x) \geq 0$

$$\therefore \quad \frac{\cos x - \sin x}{2+2\sin x\cos x} \geq 0$$

$$\Rightarrow \quad \cos x - \sin x \geq 0 \quad \left[\because \ (2+\sin 2x) \geq 0 \text{ in } \left(0, \frac{\pi}{4}\right)\right]$$

$\Rightarrow \quad \cos x \geq \sin x$, which is true for $\left(0, \frac{\pi}{4}\right)$

Hence, the given function $f(x)$ is an increasing function in $\left(0, \frac{\pi}{4}\right)$.

Q23. At what point, the slope of the curve $y = -x^3 + 3x^2 + 9x - 27$ is maximum ? Also find the maximum slope.

Sol. Given that: $y = -x^3 + 3x^2 + 9x - 27$

Differentiating both sides w.r.t. x, we get $\frac{dy}{dx} = -3x^2 + 6x + 9$

Let slope of the cuve $\frac{dy}{dx} = Z$

$$\therefore \quad z = -3x^2 + 6x + 9$$

Differentiating both sides w.r.t. x, we get $\frac{dz}{dx} = -6x + 6$

For local maxima and local minima, $\frac{dz}{dx} = 0$

$$\therefore \quad -6x + 6 = 0 \quad \Rightarrow \quad x = 1$$

$$\Rightarrow \quad \frac{d^2z}{dx^2} = -6 < 0 \quad \text{Maxima}$$

Put $x = 1$ in equation of the curve $y = (-1)^3 + 3(1)^2 + 9(1) - 27$

$$= -1 + 3 + 9 - 27 = -16$$

Maximum slope $= -3(1)^2 + 6(1) + 9 = 12$

Hence, $(1, -16)$ is the point at which the slope of the given curve is maximum and maximum slope = 12.

Q24. Prove that $f(x) = \sin x + \sqrt{3}\cos x$ has maximum value at $x = \frac{\pi}{6}$.

Sol. We have: $f(x) = \sin x + \sqrt{3}\cos x = 2\left(\frac{1}{2}\sin x + \frac{\sqrt{3}}{2}\cos x\right)$

$$= 2\left(\cos\frac{\pi}{3}\sin x + \sin\frac{\pi}{3}\cos x\right) = 2\sin\left(x + \frac{\pi}{3}\right)$$

$$f'(x) = 2\cos\left(x + \frac{\pi}{3}\right);\ f''(x) = -2\sin\left(x + \frac{\pi}{3}\right)$$

$$f''(x)_{x=\frac{\pi}{6}} = -2\sin\left(\frac{\pi}{6} + \frac{\pi}{3}\right)$$

$$= -2\sin\frac{\pi}{2} = -2.1 = -2 < 0 \text{ (Maxima)}$$

$$= -2 \times \frac{\sqrt{3}}{2} = -\sqrt{3} < 0 \text{ (Maxima)}$$

Maximum value of the function at $x = \frac{\pi}{6}$ is

$$\sin\frac{\pi}{6} + \sqrt{3}\cos\frac{\pi}{6} = \frac{1}{2} + \sqrt{3}\,.\,\frac{\sqrt{3}}{2} = 2$$

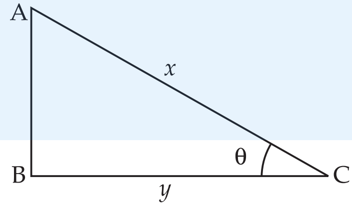
Hence, the given function has maximum value at $x = \frac{\pi}{6}$ and the maximum value is 2.

LONG ANSWER TYPE QUESTIONS

Q25. If the sum of the lengths of the hypotenuse and a side of a right angled triangle is given, show that the area of the triangle is maximum when the angle between them is $\frac{\pi}{3}$.

Sol. Let ΔABC be the right angled triangle in which $\angle B = 90°$

Let $AC = x$, $BC = y$



$\therefore \quad AB = \sqrt{x^2 - y^2}$

$\angle ACB = \theta$

Let $\quad Z = x + y \quad$ (given)

Now area of ΔABC, $A = \frac{1}{2} \times AB \times BC$

$\Rightarrow \quad A = \frac{1}{2} y \cdot \sqrt{x^2 - y^2} \quad \Rightarrow \quad A = \frac{1}{2} y \cdot \sqrt{(Z - y)^2 - y^2}$

Squaring both sides, we get

$$A^2 = \frac{1}{4} y^2 \left[(Z - y)^2 - y^2\right] \Rightarrow A^2 = \frac{1}{4} y^2 [Z^2 + y^2 - 2Zy - y^2]$$

$$\Rightarrow \quad P = \frac{1}{4} y^2 [Z^2 - 2Zy] \Rightarrow P = \frac{1}{4} [y^2 Z^2 - 2Zy^3] \qquad [A^2 = P]$$

Differentiating both sides w.r.t. y we get

$$\frac{dP}{dy} = \frac{1}{4} [2yZ^2 - 6Zy^2] \qquad \ldots(i)$$

For local maxima and local minima, $\frac{dP}{dy} = 0$

$\therefore \quad \frac{1}{4}(2yZ^2 - 6Zy^2) = 0$

$\Rightarrow \quad \frac{2yZ}{4}(Z - 3y) = 0 \quad \Rightarrow \quad yZ(Z - 3y) = 0$

$\Rightarrow \quad yZ \neq 0 \qquad (\because \quad y \neq 0 \text{ and } Z \neq 0)$

$\therefore \quad Z - 3y = 0$

$\Rightarrow \quad y = \frac{Z}{3} \quad \Rightarrow y = \frac{x + y}{3} \qquad (\because \quad Z = x + y)$

$\Rightarrow \quad 3y = x + y \quad \Rightarrow 3y - y = x \quad \Rightarrow 2y = x$

$\Rightarrow \quad \frac{y}{x} = \frac{1}{2} \quad \Rightarrow \cos\theta = \frac{1}{2}$

$\therefore \quad \theta = \frac{\pi}{3}$

Differentiating eq. (i) w.r.t. y, we have $\frac{d^2P}{dy^2} = \frac{1}{4}[2Z^2 - 12Zy]$

$$\frac{d^2P}{dy^2} \text{ at } y = \frac{Z}{3} = \frac{1}{4}\left[2Z^2 - 12Z \cdot \frac{Z}{3}\right]$$

$$= \frac{1}{4}[2Z^2 - 4Z^2] = \frac{-Z^2}{2} < 0 \text{ Maxima}$$

Hence, the area of the given triangle is maximum when the angle between its hypotenuse and a side is $\frac{\pi}{3}$.

Q26. Find the points of local maxima, local minima and the points of inflection of the function $f(x) = x^5 - 5x^4 + 5x^3 - 1$. Also find the corresponding local maximum and local minimum values.

Sol. We have $\quad f(x) = x^5 - 5x^4 + 5x^3 - 1$

$\Rightarrow \quad f'(x) = 5x^4 - 20x^3 + 15x^2$

For local maxima and local minima, $f'(x) = 0$

$\Rightarrow \quad 5x^4 - 20x^3 + 15x^2 = 0 \quad \Rightarrow 5x^2(x^2 - 4x + 3) = 0$

$\Rightarrow \quad 5x^2(x^2 - 3x - x + 3) = 0 \quad \Rightarrow x^2(x - 3)(x - 1) = 0$

$\therefore \quad x = 0, x = 1$ and $x = 3$

Now $\quad f''(x) = 20x^3 - 60x^2 + 30x$

$\Rightarrow \quad f''(x)_{\text{at } x=0} = 20(0)^3 - 60(0)^2 + 30(0) = 0$ which is neither maxima nor minima.

$\therefore \quad f(x)$ has the point of inflection at $x = 0$

$$f''(x)_{\text{at } x=1} = 20(1)^3 - 60(1)^2 + 30(1)$$
$$= 20 - 60 + 30 = -10 < 0 \text{ Maxima}$$
$$f''(x)_{\text{at } x=3} = 20(3)^3 - 60(3)^2 + 30(3)$$
$$= 540 - 540 + 90 = 90 > 0 \text{ Minima}$$

The maximum value of the function at $x = 1$

$$f(x) = (1)^5 - 5(1)^4 + 5(1)^3 - 1$$
$$= 1 - 5 + 5 - 1 = 0$$

The minimum value at $x = 3$ is

$$f(x) = (3)^5 - 5(3)^4 + 5(3)^3 - 1$$
$$= 243 - 405 + 135 - 1 = 378 - 406 = -28$$

Hence, the function has its maxima at $x = 1$ and the maximum value = 0 and it has minimum value at x = 3 and its minimum value is – 28.

$x = 0$ is the point of inflection.

Q27. A telephone company in a town has 500 subscribers on its list and collects fixed charges of ₹ 300 per subscriber per year. The company proposes to increase the annual subscription and it is believed that for every increase of ₹ 1.00, one subscriber will discontinue the service. Find what increase will bring maximum profit?

Sol. Let us consider that the company increases the annual subscription by ₹ x.

So, x is the number of subscribers who discontinue the services.

$\therefore$ Total revenue, $R(x) = (500 - x)(300 + x)$

$$= 150000 + 500x - 300x - x^2$$
$$= -x^2 + 200x + 150000$$

Differentiating both sides w.r.t. x, we get $\quad R'(x) = -2x + 200$

For local maxima and local minima, $R'(x) = 0$

$$-2x + 200 = 0 \quad \Rightarrow \quad x = 100$$
$$R''(x) = -2 < 0 \text{ Maxima}$$

So, $R(x)$ is maximum at $x = 100$

Hence, in order to get maximum profit, the company should increase its annual subscription by ₹ 100.

Q28. If the straight line $x \cos \alpha + y \sin \alpha = p$ touches the curve $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then prove that $a^2 \cos^2 \alpha + b^2 \sin^2 \alpha = p^2$.

Sol. The given curve is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ...(*i*)

and the straight line $x \cos \alpha + y \sin \alpha = p$ ...(*ii*)

Differentiating eq. (*i*) w.r.t. x, we get

$$\frac{1}{a^2} \cdot 2x + \frac{1}{b^2} \cdot 2y \cdot \frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{x}{a^2} + \frac{y}{b^2} \frac{dy}{dx} = 0 \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{b^2}{a^2} \cdot \frac{x}{y}$$

So the slope of the curve $= \frac{-b^2}{a^2} \cdot \frac{x}{y}$

Now differentiating eq. (*ii*) w.r.t. x, we have

$$\cos \alpha + \sin \alpha \cdot \frac{dy}{dx} = 0$$

$$\therefore \quad \frac{dy}{dx} = \frac{-\cos \alpha}{\sin \alpha} = -\cot \alpha$$

So, the slope of the straight line = – cot α

If the line is the tangent to the curve, then

$$\frac{-b^2}{a^2} \cdot \frac{x}{y} = -\cot \alpha \quad \Rightarrow \quad \frac{x}{y} = \frac{a^2}{b^2} \cdot \cot \alpha \quad \Rightarrow \quad x = \frac{a^2}{b^2} \cot \alpha \cdot y$$

Now from eq. (*ii*) we have $x \cos \alpha + y \sin \alpha = p$

$$\Rightarrow \quad \frac{a^2}{b^2} \cdot \cot \alpha \cdot y \cdot \cos \alpha + y \sin \alpha = p$$

$$\Rightarrow \quad a^2 \cot \alpha \cdot \cos \alpha \, y + b^2 \sin \alpha \, y = b^2 p$$

$$\Rightarrow \quad a^2 \frac{\cos \alpha}{\sin \alpha} \cdot \cos \alpha \, y + b^2 \sin \alpha \, y = b^2 p$$

$$\Rightarrow \quad a^2 \cos^2 \alpha \, y + b^2 \sin^2 \alpha \, y = b^2 \sin \alpha \, p$$

$$\Rightarrow \quad a^2 \cos^2 \alpha + b^2 \sin^2 \alpha = \frac{b^2}{y} \cdot \sin \alpha \cdot p$$

$$\Rightarrow \quad a^2 \cos^2 \alpha + b^2 \sin^2 \alpha = p \cdot p \quad \left[\because \frac{b^2}{y} \sin \alpha = p\right]$$

Hence, $a^2 \cos^2 \alpha + b^2 \sin^2 \alpha = p^2$

Alternate method:

We know that $y = mx + c$ will touch the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ if } c^2 = a^2m^2 + b^2$$

Here equation of straight line is $x \cos \alpha + y \sin \alpha = p$ and that of ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$x \cos \alpha + y \sin \alpha = p$$

$$\Rightarrow \quad y \sin \alpha = -x \cos \alpha + p$$

$$\Rightarrow \quad y = -x \frac{\cos \alpha}{\sin \alpha} + \frac{p}{\sin \alpha} \quad \Rightarrow y = -x \cot \alpha + \frac{p}{\sin \alpha}$$

Comparing with $y = mx + c$, we get

$$m = -\cot \alpha \quad \text{and} \quad c = \frac{p}{\sin \alpha}$$

So, according to the condition, we get $c^2 = a^2m^2 + b^2$

$$\frac{p^2}{\sin^2 \alpha} = a^2(-\cot \alpha)^2 + b^2$$

$$\Rightarrow \quad \frac{p^2}{\sin^2 \alpha} = \frac{a^2 \cos^2 \alpha}{\sin^2 \alpha} + b^2 \quad \Rightarrow p^2 = a^2 \cos^2 \alpha + b^2 \sin^2 \alpha$$

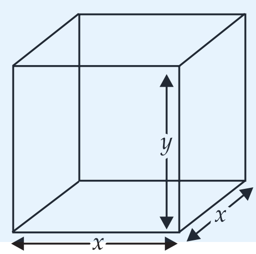
Hence, $a^2 \cos^2 \alpha + b^2 \sin^2 \alpha = p^2$ Hence proved.

Q29. An open box with square base is to be made of a given quantity of card board of area c^2. Show that the maximum volume of the box is $\frac{c^3}{6\sqrt{3}}$ cubic units.

Sol. Let x be the length of the side of the square base of the cubical open box and y be its height.

$\therefore$ Surface area of the open box

$$c^2 = x^2 + 4xy \quad \Rightarrow y = \frac{c^2 - x^2}{4x} \qquad \ldots(i)$$



Now volume of the box, $V = x \times x \times y$

$$\Rightarrow \quad V = x^2y$$

$$\Rightarrow \quad V = x^2\left(\frac{c^2 - x^2}{4x}\right)$$

$$\Rightarrow \quad V = \frac{1}{4}(c^2x - x^3)$$

Differentiating both sides w.r.t. x, we get

$$\frac{dV}{dx} = \frac{1}{4}(c^2 - 3x^2) \qquad ...(ii)$$

For local maxima and local minima, $\frac{dV}{dx} = 0$

$\therefore \quad \frac{1}{4}(c^2 - 3x^2) = 0 \Rightarrow c^2 - 3x^2 = 0$

$\Rightarrow \qquad x^2 = \frac{c^2}{3}$

$\therefore \qquad x = \sqrt{\frac{c^2}{3}} = \frac{c}{\sqrt{3}}$

Now again differentiating eq. (*ii*) w.r.t. x, we get

$$\frac{d^2V}{dx^2} = \frac{1}{4}(-6x) = \frac{-3}{2} \cdot \frac{c}{\sqrt{3}} < 0 \quad \text{(maxima)}$$

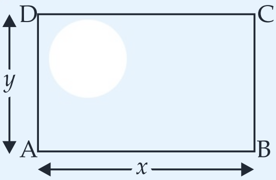
Volume of the cubical box (V) = $x^2 y$

$$= x^2\left(\frac{c^2 - x^2}{4x}\right) = \frac{c}{\sqrt{3}}\left[\frac{c^2 - \frac{c^2}{3}}{4}\right] = \frac{c}{\sqrt{3}} \times \frac{2c^2}{3 \times 4} = \frac{c^3}{6\sqrt{3}}$$

Hence, the maximum volume of the open box is $\frac{c^3}{6\sqrt{3}}$ cubic units.

Q30. Find the dimensions of the rectangle of perimeter 36 cm which will sweep out a volume as large as possible, when revolved about one of its sides. Also find the maximum volume.

Sol. Let x and y be the length and breadth of a given rectangle ABCD as per question, the rectangle be revolved about side AD which will make a cylinder with radius x and height y.



$\therefore \quad$ Volume of the cylinder $V = \pi r^2 h$

$\Rightarrow \qquad V = \pi x^2 y \qquad ...(i)$

Now perimeter of rectangle $P = 2(x + y) \Rightarrow 36 = 2(x + y)$

$\Rightarrow \qquad x + y = 18 \Rightarrow y = 18 - x \qquad ...(ii)$

Putting the value of y in eq. (*i*) we get

$V = \pi x^2(18 - x)$

$\Rightarrow \qquad V = \pi(18x^2 - x^3)$

Differentiating both sides w.r.t. x, we get

$$\frac{dV}{dx} = \pi(36x - 3x^2) \qquad ...(iii)$$

For local maxima and local minima $\frac{dV}{dx} = 0$

$\therefore \quad \pi(36x - 3x^2) = 0 \quad \Rightarrow 36x - 3x^2 = 0$

$\Rightarrow \quad 3x(12 - x) = 0$

$\Rightarrow \quad x \neq 0 \quad \therefore \quad 12 - x = 0 \Rightarrow x = 12$

From eq. (*ii*) $y = 18 - 12 = 6$

Differentiating eq. (*iii*) w.r.t. x, we get $\frac{d^2V}{dx^2} = \pi(36 - 6x)$

at $x = 12 \quad \frac{d^2V}{dx^2} = \pi(36 - 6 \times 12)$

$= \pi(36 - 72) = -36\pi < 0$ maxima

Now volume of the cylinder so formed $= \pi x^2 y$

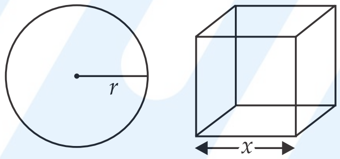
$= \pi \times (12)^2 \times 6 = \pi \times 144 \times 6 = 864\pi \text{ cm}^3$

Hence, the required dimensions are 12 cm and 6 cm and the maximum volume is 864π cm^3.

Q31. If the sum of the surface areas of cube and a sphere is constant, what is the ratio of an edge of the cube to the diameter of the sphere, when the sum of their volumes is minimum?

Sol. Let x be the edge of the cube and r be the radius of the sphere.

Surface area of cube $= 6x^2$



and surface area of the sphere $= 4\pi r^2$

$\therefore \quad 6x^2 + 4\pi r^2 = K(\text{constant}) \quad \Rightarrow \quad r = \sqrt{\frac{K - 6x^2}{4\pi}} \quad ...(i)$

Volume of the cube $= x^3$ and the volume of sphere $= \frac{4}{3}\pi r^3$

$\therefore$ Sum of their volumes (V) = Volume of cube + Volume of sphere

$\Rightarrow \quad V = x^3 + \frac{4}{3}\pi r^3$

$\Rightarrow \quad V = x^3 + \frac{4}{3}\pi \times \left(\frac{K - 6x^2}{4\pi}\right)^{3/2}$

Differentiating both sides w.r.t. x, we get

$$\frac{dV}{dx} = 3x^2 + \frac{4\pi}{3} \times \frac{3}{2}(K - 6x^2)^{1/2}(-12x) \times \frac{1}{(4\pi)^{3/2}}$$

$$= 3x^2 + \frac{2\pi}{(4\pi)^{3/2}} \times (-12x)\,(K - 6x^2)^{1/2}$$

$$= 3x^2 + \frac{1}{4\pi^{1/2}} \times (-12x)\,(K - 6x^2)^{1/2}$$

$$\therefore \quad \frac{dV}{dx} = 3x^2 - \frac{3x}{\sqrt{\pi}}(K - 6x^2)^{1/2} \qquad \text{...}(ii)$$

For local maxima and local minima, $\frac{dV}{dx} = 0$

$$\therefore \quad 3x^2 - \frac{3x}{\sqrt{\pi}}(K - 6x^2)^{1/2} = 0$$

$$\Rightarrow \quad 3x\left[x - \frac{(K - 6x^2)^{1/2}}{\sqrt{\pi}}\right] = 0$$

$$x \neq 0 \quad \therefore \quad x - \frac{(K - 6x^2)^{1/2}}{\sqrt{\pi}} = 0$$

$$\Rightarrow \quad x = \frac{(K - 6x^2)^{1/2}}{\sqrt{\pi}}$$

Squaring both sides, we get

$$x^2 = \frac{K - 6x^2}{\pi} \quad \Rightarrow \pi x^2 = K - 6x^2$$

$$\Rightarrow \quad \pi x^2 + 6x^2 = K \quad \Rightarrow x^2(\pi + 6) = K \quad \Rightarrow x^2 = \frac{K}{\pi + 6}$$

$$\therefore \quad x = \sqrt{\frac{K}{\pi + 6}}$$

Now putting the value of K in eq. (*i*), we get

$$6x^2 + 4\pi r^2 = x^2(\pi + 6)$$

$$\Rightarrow \quad 6x^2 + 4\pi r^2 = \pi x^2 + 6x^2 \quad \Rightarrow 4\pi r^2 = \pi x^2 \quad \Rightarrow 4r^2 = x^2$$

$$\therefore \quad 2r = x$$

$$\therefore \quad x : 2r = 1 : 1$$

Now differentiating eq. (*ii*) w.r.t x, we have

$$\frac{d^2V}{dx^2} = 6x - \frac{3}{\sqrt{\pi}} \frac{d}{dx}[x(K - 6x^2)^{1/2}]$$

$$= 6x - \frac{3}{\sqrt{\pi}}\left[x \cdot \frac{1}{2\sqrt{K - 6x^2}} \times (-12x) + (K - 6x^2)^{1/2} \cdot 1\right]$$

$$= 6x - \frac{3}{\sqrt{\pi}}\left[\frac{-6x^2}{\sqrt{K - 6x^2}} + \sqrt{K - 6x^2}\right]$$

$$= 6x - \frac{3}{\sqrt{\pi}}\left[\frac{-6x^2 + K - 6x^2}{\sqrt{K - 6x^2}}\right] = 6x + \frac{3}{\sqrt{\pi}}\left[\frac{12x^2 - K}{\sqrt{K - 6x^2}}\right]$$

Put $x = \sqrt{\frac{K}{\pi + 6}} = 6\sqrt{\frac{K}{\pi + 6}} + \frac{3}{\sqrt{\pi}}\left[\frac{\frac{12K}{\pi + 6} - K}{\sqrt{K - \frac{6K}{\pi + 6}}}\right]$

$$= 6\sqrt{\frac{K}{\pi + 6}} + \frac{3}{\sqrt{\pi}}\left[\frac{12K - \pi K - 6K}{\sqrt{\frac{\pi K + 6K - 6K}{\pi + 6}}}\right]$$

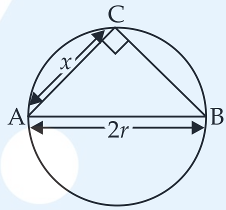
$$= 6\sqrt{\frac{K}{\pi + 6}} + \frac{3}{\sqrt{\pi}}\left[\frac{6K - \pi K}{\sqrt{\frac{\pi K}{\pi + 6}}}\right]$$

$$= 6\sqrt{\frac{K}{\pi + 6}} + \frac{3}{\pi\sqrt{K}}[(6K - \pi K)\sqrt{\pi + 6}] > 0$$

So it is minima.

Hence, the required ratio is 1 : 1 when the combined volume is minimum.

Q32. AB is a diameter of a circle and C is any point on the circle. Show that the area of ΔABC is maximum, when it is isosceles.



Sol. Let AB be the diameter and C be any point on the circle with radius r.

$\angle ACB = 90°$ [angle in the semi circle is 90°]

Let $AC = x$

$\therefore \quad BC = \sqrt{AB^2 - AC^2}$

$\Rightarrow \quad BC = \sqrt{(2r)^2 - x^2} \quad \Rightarrow BC = \sqrt{4r^2 - x^2}$ ...(i)

Now area of ΔABC, $A = \frac{1}{2} \times AC \times BC$

$\Rightarrow \quad A = \frac{1}{2} x \cdot \sqrt{4r^2 - x^2}$

Squaring both sides, we get

$$A^2 = \frac{1}{4}x^2(4r^2 - x^2)$$

Let $A^2 = Z$

$\therefore \quad Z = \frac{1}{4}x^2(4r^2 - x^2) \quad \Rightarrow Z = \frac{1}{4}(4x^2r^2 - x^4)$

Differentiating both sides w.r.t. x, we get

$$\frac{dZ}{dx} = \frac{1}{4}[8xr^2 - 4x^3] \quad ...(ii)$$

For local maxima and local minima $\frac{dZ}{dx} = 0$

$\therefore \quad \frac{1}{4}[8xr^2 - 4x^3] = 0 \Rightarrow x[2r^2 - x^2] = 0$

$x \neq 0 \quad \therefore \quad 2r^2 - x^2 = 0$

$\Rightarrow \quad x^2 = 2r^2 \Rightarrow x = \sqrt{2}r = AC$

Now from eq. (i) we have

$$BC = \sqrt{4r^2 - 2r^2} \Rightarrow BC = \sqrt{2r^2} \Rightarrow BC = \sqrt{2}r$$

So AC = BC

Hence, ΔABC is an isosceles triangle.

Differentiating eq. (ii) w.r.t. x, we get $\frac{d^2Z}{dx^2} = \frac{1}{4}[8r^2 - 12x^2]$

Put $x = \sqrt{2}r$

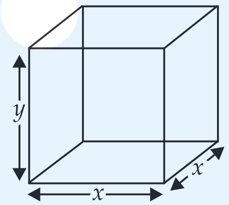
$$\therefore \quad \frac{d^2Z}{dx^2} = \frac{1}{4}[8r^2 - 12 \times 2r^2] = \frac{1}{4}[8r^2 - 24r^2]$$

$$= \frac{1}{4} \times (-16r^2) = -4r^2 < 0 \quad \text{maxima}$$

Hence, the area of ΔABC is maximum when it is an isosceles triangle.

Q33. A metal box with a square base and vertical sides is to contain 1024 cm^3. The material for the top and botttom costs ₹ 5/cm^2 and the material for the sides costs ₹ 2.50/cm^2. Find the least cost of the box.

Sol. Let x be the side of the square base and y be the length of the vertical sides.



Area of the base and bottom = $2x^2$ cm^2

$\therefore$ Cost of the material required = ₹ $5 \times 2x^2$

= ₹ $10x^2$

Area of the 4 sides = $4xy$ cm^2

$\therefore$ Cost of the material for the four sides

= ₹ $2.50 \times 4xy$ = ₹ $10xy$

Total cost $\quad C = 10x^2 + 10xy \quad ...(i)$

New volume of the box = $x \times x \times y$

$\Rightarrow \quad 1024 = x^2y$

$$\therefore \quad y = \frac{1024}{x^2} \quad ...(ii)$$

Putting the value of y in eq. (i) we get

$$C = 10x^2 + 10x \times \frac{1024}{x^2} \quad \Rightarrow C = 10x^2 + \frac{10240}{x}$$

Differentiating both sides w.r.t. x, we get

$$\frac{dC}{dx} = 20x - \frac{10240}{x^2} \qquad ...(iii)$$

For local maxima and local minima $\frac{dC}{dx} = 0$

$$20 - \frac{10240}{x^2} = 0$$

$\Rightarrow \quad 20x^3 - 10240 = 0 \quad \Rightarrow x^3 = 512 \quad \Rightarrow \quad x = 8$ cm

Now from eq. (ii)

$$y = \frac{10240}{(8)^2} = \frac{10240}{64} = 16 \text{ cm}$$

$\therefore$ Cost of material used $C = 10x^2 + 10xy$

$$= 10 \times 8 \times 8 + 10 \times 8 \times 16 = 640 + 1280 = 1920$$

Now differentiating eq. (iii) we get

$$\frac{d^2C}{dx^2} = 20 + \frac{20480}{x^3}$$

Put $x = 8$

$$= 20 + \frac{20480}{(8)^3} = 20 + \frac{20480}{512} = 20 + 40 = 60 > 0 \text{ minima}$$

Hence, the required cost is ₹ 1920 which is the minimum.

Q34. The sum of the surface areas of a rectangular parallelopiped with sides x, $2x$ and $\frac{x}{3}$ and a sphere is given to be constant. Prove that the sum of their volumes is minimum, if x is equal to three times the radius of the sphere. Also find the minimum value of the sum of their volumes.

Sol. Let 'r' be the radius of the sphere.

$\therefore$ Surface area of the sphere $= 4\pi r^2$

Volume of the sphere $= \frac{4}{3}\pi r^3$

The sides of the parallelopiped are x, $2x$ and $\frac{x}{3}$

$$\therefore \quad \text{Its surface area} = 2\left[x \times 2x + 2x \times \frac{x}{3} + x \times \frac{x}{3}\right]$$

$$= 2\left[2x^2 + \frac{2x^2}{3} + \frac{x^2}{3}\right] = 2[2x^2 + x^2]$$

$$= 2[3x^2] = 6x^2$$

Volume of the parallelopiped = $x \times 2x \times \frac{x}{3} = \frac{2}{3}x^3$

As per the conditions of the question,

Surface area of the parallelopiped

+ Surface area of the sphere = constant

$\Rightarrow \quad 6x^2 + 4\pi r^2 = \text{K (constant)} \quad \Rightarrow 4\pi r^2 = \text{K} - 6x^2$

$\therefore \quad r^2 = \frac{\text{K} - 6x^2}{4\pi}$ ...(*i*)

Now let V = Volume of parallelopiped + Volume of the sphere

$\Rightarrow \quad V = \frac{2}{3}x^3 + \frac{4}{3}\pi r^3$

$\Rightarrow \quad V = \frac{2}{3}x^3 + \frac{4}{3}\pi\left[\frac{\text{K} - 6x^2}{4\pi}\right]^{3/2}$ [from eq. (*i*)]

$\Rightarrow \quad V = \frac{2}{3}x^3 + \frac{4}{3}\pi \times \frac{1}{(4)^{3/2}\pi^{3/2}}[\text{K} - 6x^2]^{3/2}$

$\Rightarrow \quad V = \frac{2}{3}x^3 + \frac{4}{3}\pi \times \frac{1}{8 \times \pi^{3/2}}[\text{K} - 6x^2]^{3/2}$

$\Rightarrow \quad = \frac{2}{3}x^3 + \frac{1}{6\sqrt{\pi}}[\text{K} - 6x^2]^{3/2}$

Differentiating both sides w.r.t. x, we have

$$\frac{dV}{dx} = \frac{2}{3}\cdot 3x^2 + \frac{1}{6\sqrt{\pi}}\left[\frac{3}{2}(K - 6x^2)^{1/2}(-12x)\right]$$

$$= 2x^2 + \frac{1}{6\sqrt{\pi}} \times \frac{3}{2} \times (-12x)(\text{K} - 6x^2)^{1/2}$$

$$= 2x^2 - \frac{3x}{\sqrt{\pi}}[\text{K} - 6x^2)^{1/2}$$

For local maxima and local minima, we have $\frac{dV}{dx} = 0$

$\therefore \quad 2x^2 - \frac{3x}{\sqrt{\pi}}(\text{K} - 6x^2)^{1/2} = 0$

$\Rightarrow \quad 2\sqrt{\pi}x^2 - 3x(\text{K} - 6x^2)^{1/2} = 0$

$\Rightarrow \quad x[2\sqrt{\pi}x - 3(\text{K} - 6x^2)^{1/2}] = 0$

Here $x \neq 0$ and $2\sqrt{\pi}x - 3(\text{K} - 6x^2)^{1/2} = 0$

$\Rightarrow \quad 2\sqrt{\pi}x = 3(\text{K} - 6x^2)^{1/2}$

Squaring both sides, we get

$$4\pi x^2 = 9(\text{K} - 6x^2) \Rightarrow 4\pi x^2 = 9\text{K} - 54x^2$$

$$\Rightarrow \quad 4\pi x^2 + 54x^2 = 9K$$

$$\Rightarrow \quad K = \frac{4\pi x^2 + 54x^2}{9} \quad ...(ii)$$

$$\Rightarrow \quad 2x^2(2\pi + 27) = 9K$$

$$\therefore \quad x^2 = \frac{9K}{2(2\pi + 27)} = 3\sqrt{\frac{K}{4\pi + 54}}$$

Now from eq. (*i*) we have

$$r^2 = \frac{K - 6x^2}{4\pi}$$

$$\Rightarrow \quad r^2 = \frac{\frac{4\pi x^2 + 54x^2}{9} - 6x^2}{4\pi}$$

$$\Rightarrow \quad r^2 = \frac{4\pi x^2 + 54x^2 - 54x^2}{9 \times 4\pi} = \frac{4\pi x^2}{9 \times 4\pi}$$

$$\Rightarrow \quad r^2 = \frac{x^2}{9} \quad \Rightarrow \quad r = \frac{x}{3} \quad \therefore \quad x = 3r$$

Now we have $\frac{dV}{dx} = 2x^2 - \frac{3x}{\sqrt{\pi}}(K - 6x^2)^{1/2}$

Differentiating both sides w.r.t. x, we get

$$\frac{d^2V}{dx^2} = 4x - \frac{3}{\sqrt{\pi}}\left[x \cdot \frac{d}{dx}(K - 6x^2)^{1/2} + (K - 6x^2)^{1/2} \cdot \frac{d}{dx} \cdot x\right]$$

$$= 4x - \frac{3}{\sqrt{\pi}}\left[x \cdot \frac{1 \times (-12x)}{2\sqrt{K - 6x^2}} + (K - 6x^2)^{1/2} \cdot 1\right]$$

$$= 4x - \frac{3}{\sqrt{\pi}}\left[\frac{-6x^2}{(K - 6x^2)^{1/2}} + (K - 6x^2)^{1/2}\right]$$

$$= 4x - \frac{3}{\sqrt{\pi}}\left[\frac{-6x^2 + K - 6x^2}{(K - 6x^2)^{1/2}}\right] = 4x - \frac{3}{\sqrt{\pi}}\left[\frac{K - 12x^2}{(K - 6x^2)^{1/2}}\right]$$

Put $x = 3 \cdot \sqrt{\frac{K}{4\pi + 54}}$

$$\frac{d^2V}{dx^2} = 4 \cdot 3\sqrt{\frac{K}{4\pi + 54}} - \frac{3}{\sqrt{\pi}}\left[\frac{K - 12 \cdot \frac{9K}{4\pi + 54}}{\sqrt{(K - 6 \cdot \frac{9K}{4\pi + 54})}}\right]$$

$$= 12\sqrt{\frac{K}{4\pi+54}} - \frac{3}{\sqrt{\pi}}\frac{\frac{4K\pi+54K-108K}{4\pi+54}}{\sqrt{\frac{4K\pi+54K-54K}{4\pi+54}}}$$

$$= 12\sqrt{\frac{K}{4\pi+54}} - \frac{3}{\sqrt{\pi}}\left[\frac{\frac{4K\pi-54K}{4\pi+54}}{\sqrt{\frac{4K\pi}{4\pi+54}}}\right]$$

$$= 12\sqrt{\frac{K}{4\pi+54}} - \frac{3}{\sqrt{\pi}}\left[\frac{4K\pi-54K}{\sqrt{4K\pi}\cdot\sqrt{4\pi+54}}\right]$$

$$= 12\sqrt{\frac{K}{4\pi+54}} - \frac{6K}{\sqrt{\pi}}\left(\frac{2\pi-27}{\sqrt{4K\pi}\cdot\sqrt{4\pi+54}}\right)$$

$$= 12\sqrt{\frac{K}{4\pi+54}} + \frac{6K}{\sqrt{\pi}}\left[\frac{27-2\pi}{\sqrt{4k\pi}\cdot\sqrt{4\pi+54}}\right] > 0$$

$[\because \quad 27 - 2\pi > 0]$

$\therefore \quad \frac{d^2V}{dx^2} > 0$ so, it is minima.

Hence, the sum of volume is minimum for $x = 3\sqrt{\frac{K}{4\pi+54}}$

$\therefore$ Minimum volume,

$$V \text{ at } \left(x = 3\sqrt{\frac{K}{4\pi+54}}\right) = \frac{2}{3}x^3 + \frac{4}{3}\pi r^3 = \frac{2}{3}x^3 + \frac{4}{3}\pi\cdot\left(\frac{x}{3}\right)^3$$

$$= \frac{2}{3}x^3 + \frac{4}{3}\pi\cdot\frac{x^3}{27} = \frac{2}{3}x^3 + \frac{4}{81}\pi x^3$$

$$= \frac{2}{3}x^3\left(1+\frac{2\pi}{27}\right)$$

Hence, the required minimum volume is $\frac{2}{3}x^3\left(1+\frac{2\pi}{27}\right)$ and $x = 3r$.

OBJECTIVE TYPE QUESTIONS

Choose the correct answer from the given four options in each of the following questions 35 to 59:

Q35. The sides of an equilateral triangle are increasing at the rate of 2 cm/sec. The rate at which the area increases, when side is 10 cm is:

(a) $10 \text{ cm}^2/\text{s}$ (b) $\sqrt{3} \text{ cm}^2/\text{s}$

(c) $10\sqrt{3} \text{ cm}^2/\text{s}$ (d) $\frac{10}{3} \text{ cm}^2/\text{s}$

Sol. Let the length of each side of the given equilateral triangle be x cm.

$\therefore \quad \frac{dx}{dt} = 2$ cm/sec

Area of equilateral triangle $A = \frac{\sqrt{3}}{4}x^2$

$$\therefore \quad \frac{dA}{dt} = \frac{\sqrt{3}}{4} \cdot 2x \cdot \frac{dx}{dt} = \frac{\sqrt{3}}{2} \times 10 \times 2 = 10\sqrt{3} \text{ cm}^2/\text{sec}$$

Hence, the rate of increasing of area = $10\sqrt{3} \text{ cm}^2/\text{sec}$.

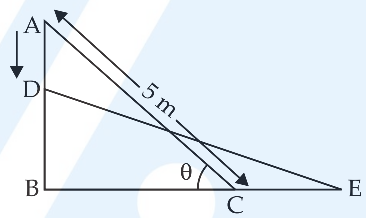
Hence, the correct option is (c).

Q36. A ladder, 5 m long, standing on a horizontal floor, leans against a vertical wall. If the top of the ladder slides downwards at the rate of 10 cm/sec, then the rate at which the angle between the floor and the ladder is decreasing when lower end of ladder is 2 metres from the wall is:

(a) $\frac{1}{10}$ radian/sec (b) $\frac{1}{20}$ radian/sec

(c) 20 radian/sec

(d) 10 radian/sec



Sol. Length of ladder = 5 m

Let AB = y m and BC = x m

$\therefore$ In right ΔABC,

$$AB^2 + BC^2 = AC^2$$

$$\Rightarrow \quad x^2 + y^2 = (5)^2 \Rightarrow x^2 + y^2 = 25$$

Differentiating both sides w.r.t x, we have

$$2x \cdot \frac{dx}{dt} + 2y \cdot \frac{dy}{dt} = 0$$

$$\Rightarrow \quad x\frac{dx}{dt} + y \cdot \frac{dy}{dt} = 0$$

$$\Rightarrow \quad 2 \cdot \frac{dx}{dt} + y \times (-0.1) = 0 \quad [\because \; x = 2m]$$

$$\Rightarrow \quad 2 \cdot \frac{dx}{dt} + \sqrt{25 - x^2} \times (-0.1) = 0$$

$$\Rightarrow \quad 2 \cdot \frac{dx}{dt} + \sqrt{25 - 4} \times (-0.1) = 0$$

$$\Rightarrow \quad 2 \cdot \frac{dx}{dt} - \frac{\sqrt{21}}{10} = 0 \Rightarrow \frac{dx}{dt} = \frac{\sqrt{21}}{20}$$

Now $\cos\theta = \frac{BC}{AC}$ (θ is in radian)

$\Rightarrow$ $\cos\theta = \frac{x}{5}$

Differentiating both sides w.r.t. t, we get

$$\frac{d}{dt}\cos\theta = \frac{1}{5}\cdot\frac{dx}{dt} \Rightarrow -\sin\theta\cdot\frac{d\theta}{dt} = \frac{1}{5}\cdot\frac{\sqrt{21}}{20}$$

$$\Rightarrow \frac{d\theta}{dt} = \frac{\sqrt{21}}{100}\times\left(-\frac{1}{\sin\theta}\right) = \frac{\sqrt{21}}{100}\times-\left(\frac{1}{\frac{AB}{AC}}\right)$$

$$= -\frac{\sqrt{21}}{100}\times\frac{AC}{AB} = -\frac{\sqrt{21}}{100}\times\frac{5}{\sqrt{21}} = -\frac{1}{20} \text{ radian/sec}$$

[(–) sign shows the decrease of change of angle]

Hence, the required rate = $\frac{1}{20}$ radian/sec

Hence, the correct option is (*b*).

Q37. The curve $y = x^{1/5}$ has at (0, 0)

(*a*) a vertical tangent (parallel to y-axis)
(*b*) a horizontal tangent (parallel to x-axis)
(*c*) an oblique tangent
(*d*) no tangent

Sol. Equation of curve is $y = x^{1/5}$

Differentiating w.r.t. x, we get $\frac{dy}{dx} = \frac{1}{5}x^{-4/5}$

(at $x = 0$) $\frac{dy}{dx} = \frac{1}{5}(0)^{-4/5} = \frac{1}{5}\times\frac{1}{0} = \infty$

$$\frac{dy}{dx} = \infty$$

$\therefore$ The tangent is parallel to y-axis.

Hence, the correct option is (*a*).

Q38. The equation of normal to the curve $3x^2 - y^2 = 8$ which is parallel to the line $x + 3y = 8$ is

(*a*) $3x - y = 8$ (*b*) $3x + y + 8 = 0$
(*c*) $x + 3y \pm 8 = 0$ (*d*) $x + 3y = 0$

Sol. Given equation of the curve is $3x^2 - y^2 = 8$ ...(*i*)

Differentiating both sides w.r.t. x, we get

$$6x - 2y\cdot\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{3x}{y}$$

$\frac{3x}{y}$ is the slope of the tangent

$\therefore$ Slope of the normal $= \frac{-1}{dy/dx} = \frac{-y}{3x}$

Now $x + 3y = 8$ is parallel to the normal

Differentiating both sides w.r.t. x, we have

$$1 + 3\frac{dy}{dx} = 0 \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{1}{3}$$

$$\therefore \quad \frac{-y}{3x} = -\frac{1}{3} \quad \Rightarrow \quad y = x$$

Putting $y = x$ in eq. (i) we get

$$3x^2 - x^2 = 8 \quad \Rightarrow \quad 2x^2 = 8 \quad \Rightarrow \quad x^2 = 4$$

$$\therefore \quad x = \pm 2 \text{ and } y = \pm 2$$

So the points are (2, 2) and (– 2, – 2).

Equation of normal to the given curve at (2, 2) is

$$y - 2 = -\frac{1}{3}(x - 2)$$

$$\Rightarrow \quad 3y - 6 = -x + 2 \quad \Rightarrow x + 3y - 8 = 0$$

Equation of normal at (– 2, – 2) is

$$y + 2 = -\frac{1}{3}(x + 2)$$

$$\Rightarrow \quad 3y + 6 = -x - 2 \quad \Rightarrow x + 3y + 8 = 0$$

$\therefore$ The equations of the normals to the curve are

$$x + 3y \pm 8 = 0$$

Hence, the correct option is (c).

Q39. If the curve $ay + x^2 = 7$ and $x^3 = y$, cut orthogonally at (1, 1), then the value of 'a' is:

(a) 1 (b) 0 (c) – 6 (d) 6

Sol. Equation of the given curves are $ay + x^2 = 7$ ...(i)

and $x^3 = y$ ...(ii)

Differentiating eq. (i) w.r.t. x, we have

$$a\frac{dy}{dx} + 2x = 0 \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{2x}{a}$$

$$\therefore \quad m_1 = -\frac{2x}{a} \qquad \left(m_1 = \frac{dy}{dx}\right)$$

Now differentiating eq. (ii) w.r.t. x, we get

$$3x^2 = \frac{dy}{dx} \quad \Rightarrow \quad m_2 = 3x^2 \qquad \left(m_2 = \frac{dy}{dx}\right)$$

The two curves are said to be orthogonal if the angle between the tangents at the point of intersection is 90°.

$\therefore \quad m_1 \times m_2 = -1$

$\Rightarrow \quad \dfrac{-2x}{a} \times 3x^2 = -1 \quad \Rightarrow \quad \dfrac{-6x^3}{a} = -1 \quad \Rightarrow 6x^3 = a$

(1, 1) is the point of intersection of two curves.

$\therefore \quad 6(1)^3 = a$

So $\quad a = 6$

Hence, the correct option is (*d*).

Q40. If $y = x^4 - 10$ and if x changes from 2 to 1.99, what is the change in y?

(*a*) 0.32 (*b*) 0.032 (*c*) 5.68 (*d*) 5.968

Sol. Given that $y = x^4 - 10$

$$\frac{dy}{dx} = 4x^3$$

$$\Delta x = 2.00 - 1.99 = 0.01$$

$$\therefore \quad \Delta y = \frac{dy}{dx} \cdot \Delta x = 4x^3 \times \Delta x$$

$$= 4 \times (2)^3 \times 0.01 = 32 \times 0.01 = 0.32$$

Hence, the correct option is (*a*).

Q41. The equation of tangent to the curve $y(1 + x^2) = 2 - x$, where it crosses x-axis is:

(*a*) $x + 5y = 2$ (*b*) $x - 5y = 2$

(*c*) $5x - y = 2$ (*d*) $5x + y = 2$

Sol. Given that $y(1 + x^2) = 2 - x$ ...(*i*)

If it cuts x-axis, then y-coordinate is 0.

$\therefore \quad 0(1 + x^2) = 2 - x \quad \Rightarrow \quad x = 2$

Put $x = 2$ in equation (*i*)

$$y(1 + 4) = 2 - 2 \quad \Rightarrow y(5) = 0 \quad \Rightarrow \quad y = 0$$

Point of contact = (2, 0)

Differentiating eq. (*i*) w.r.t. x, we have

$$y \times 2x + (1 + x^2)\frac{dy}{dx} = -1$$

$$\Rightarrow \quad 2xy + (1 + x^2)\frac{dy}{dx} = -1 \quad \Rightarrow (1 + x^2)\frac{dy}{dx} = -1 - 2xy$$

$$\therefore \quad \frac{dy}{dx} = \frac{-(1 + 2xy)}{(1 + x^2)} \quad \Rightarrow \left.\frac{dy}{dx}\right|_{(2,0)} = \frac{-1}{(1 + 4)} = \frac{-1}{5}$$

Equation of tangent is $y - 0 = -\dfrac{1}{5}(x - 2)$

$\Rightarrow \quad 5y = -x + 2 \quad \Rightarrow \quad x + 5y = 2$

Hence, the correct option is (a).

Q42. The points at which the tangents to the curve $y = x^3 - 12x + 18$ are parallel to x-axis are:

(*a*) (2, – 2), (– 2, – 34) (*b*) (2, 34), (– 2, 0)

(*c*) (0, 34), (– 2, 0) (*d*) (2, 2), (– 2, 34)

Sol. Given that y = $x^3 - 12x + 18$

Differentiating both sides w.r.t. x, we have

$$\Rightarrow \quad \frac{dy}{dx} = 3x^2 - 12$$

Since the tangents are parallel to x-axis, then $\frac{dy}{dx} = 0$

$$\therefore \quad 3x^2 - 12 = 0 \quad \Rightarrow \quad x = \pm 2$$

$$\therefore \quad y_{x=2} = (2)^3 - 12(2) + 18 = 8 - 24 + 18 = 2$$

$$y_{x=-2} = (-2)^3 - 12(-2) + 18 = -8 + 24 + 18 = 34$$

$\therefore$ Points are (2, 2) and (– 2, 34)

Hence, the correct option is (d).

Q43. The tangent to the curve $y = e^{2x}$ at the point (0, 1) meets x-axis at:

(*a*) (0, 1) (*b*) $\left(-\frac{1}{2}, 0\right)$ (*c*) (2, 0) (*d*) (0, 2)

Sol. Equation of the curve is $y = e^{2x}$

Slope of the tangent $\frac{dy}{dx} = 2e^{2x} \Rightarrow \left.\frac{dy}{dx}\right|_{(0,1)} = 2 \cdot e^0 = 2$

$\therefore$ Equation of tangent to the curve at (0, 1) is

$$y - 1 = 2(x - 0)$$

$$\Rightarrow \quad y - 1 = 2x \quad \Rightarrow y - 2x = 1$$

Since the tangent meets x-axis where $y = 0$

$$\therefore \quad 0 - 2x = 1 \quad \Rightarrow \quad x = \frac{-1}{2}$$

So the point is $\left(-\frac{1}{2}, 0\right)$

Hence, the correct option is (b).

Q44. The slope of tangent to the curve $x = t^2 + 3t - 8$ and $y = 2t^2 - 2t - 5$ at the point (2, – 1) is:

(*a*) $\frac{22}{7}$ (*b*) $\frac{6}{7}$ (*c*) $-\frac{6}{7}$ (*d*) – 6

Sol. The given curve is $x = t^2 + 3t - 8$ and $y = 2t^2 - 2t - 5$

$$\frac{dx}{dt} = 2t + 3 \quad \text{and} \quad \frac{dy}{dt} = 4t - 2$$

$$\therefore \quad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{4t - 2}{2t + 3}$$

Now (2, – 1) lies on the curve

$\therefore \quad 2 = t^2 + 3t - 8 \Rightarrow t^2 + 3t - 10 = 0$

$\Rightarrow t^2 + 5t - 2t - 10 = 0$

$\Rightarrow t(t + 5) - 2(t + 5) = 0$

$\Rightarrow (t + 5)(t - 2) = 0$

$\therefore t = 2, t = -5$ and $-1 = 2t^2 - 2t - 5$

$\Rightarrow 2t^2 - 2t - 4 = 0$

$\Rightarrow t^2 - t - 2 = 0 \Rightarrow t^2 - 2t + t - 2 = 0$

$\Rightarrow t(t - 2) + 1(t - 2) = 0 \Rightarrow (t + 1)(t - 2) = 0$

$\Rightarrow t = -1$ and $t = 2$

So t = 2 is common value

$$\therefore \quad \text{Slope } \frac{dy}{dx}_{x=2} = \frac{4 \times 2 - 2}{2 \times 2 + 3} = \frac{6}{7}$$

Hence, the correct option is (b).

Q45. The two curves $x^3 - 3xy^2 + 2 = 0$ and $3x^2y - y^3 - 2 = 0$ intersect at an angle of:

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{2}$ (d) $\frac{\pi}{6}$

Sol. The given curves are $x^3 - 3xy^2 + 2 = 0$ ...(i)

and $3x^2y - y^3 - 2 = 0$ ...(ii)

Differentiating eq. (i) w.r.t. x, we get

$$3x^2 - 3\left(x \cdot 2y\frac{dy}{dx} + y^2 \cdot 1\right) = 0$$

$$\Rightarrow x^2 - 2xy\frac{dy}{dx} - y^2 = 0 \Rightarrow 2xy\frac{dy}{dx} = x^2 - y^2$$

$$\therefore \quad \frac{dy}{dx} = \frac{x^2 - y^2}{2xy}$$

So slope of the curve $m_1 = \dfrac{x^2 - y^2}{2xy}$

Differentiating eq. (ii) w.r.t. x, we get

$$3\left[x^2\frac{dy}{dx} + y \cdot 2x\right] - 3y^2 \cdot \frac{dy}{dx} = 0$$

$$x^2\frac{dy}{dx} + 2xy - y^2\frac{dy}{dx} = 0 \Rightarrow (x^2 - y^2)\frac{dy}{dx} = -2xy$$

$$\therefore \quad \frac{dy}{dx} = \frac{-2xy}{x^2 - y^2}$$

So the slope of the curve $m_2 = \dfrac{-2xy}{x^2 - y^2}$

Now $m_1 \times m_2 = \dfrac{x^2 - y^2}{2xy} \times \dfrac{-2xy}{x^2 - y^2} = -1$

So the angle between the curves is $\dfrac{\pi}{2}$.

Hence, the correct option is (*c*).

Q46. The interval on which the function $f(x) = 2x^3 + 9x^2 + 12x - 1$ is decreasing is:

(*a*) $[-1, \infty)$ (*b*) $[-2, -1]$

(*c*) $(-\infty, -2]$ (*d*) $[-1, 1]$

Sol. The given function is $f(x) = 2x^3 + 9x^2 + 12x - 1$

$$f'(x) = 6x^2 + 18x + 12$$

For increasing and decreasing $f'(x) = 0$

$\therefore \quad 6x^2 + 18x + 12 = 0$

$\Rightarrow \quad x^2 + 3x + 2 = 0 \quad \Rightarrow \quad x^2 + 2x + x + 2 = 0$

$\Rightarrow \quad x(x + 2) + 1(x + 2) = 0 \quad \Rightarrow \quad (x + 2)(x + 1) = 0$

$\Rightarrow \quad x = -2, x = -1$

The possible intervals are $(-\infty, -2), (-2, -1), (-1, \infty)$

Now $f'(x) = (x + 2)(x + 1)$

$\Rightarrow \quad f'(x)_{(-\infty, -2)} = (-)(-) = (+)$ increasing

$\Rightarrow \quad f'(x)_{(-2, -1)} = (+)(-) = (-)$ decreasing

$\Rightarrow \quad f'(x)_{(-1, \infty)} = (+)(+) = (+)$ increasing

Hence, the correct option is (*b*).

Q47. Let the $f : \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = 2x + \cos x$, then f:

(*a*) has a minimum at $x = \pi$ (*b*) has a maximum at $x = 0$

(*c*) is a decreasing function (*d*) is an increasing function

Sol. Given that $f(x) = 2x + \cos x$

$f'(x) = 2 - \sin x$

Since $f'(x) > 0 \ \forall \ x$

So $f(x)$ is an increasing function.

Hence, the correct option is (*d*).

Q48. $y = x(x - 3)^2$ decreases for the values of x given by:

(*a*) $1 < x < 3$ (*b*) $x < 0$ (*c*) $x > 0$ (*d*) $0 < x < \dfrac{3}{2}$

Sol. Here $y = x(x - 3)^2$

$$\frac{dy}{dx} = x \cdot 2(x - 3) + (x - 3)^2 \cdot 1 \quad \Rightarrow \frac{dy}{dx} = 2x(x - 3) + (x - 3)^2$$

For increasing and decreasing $\frac{dy}{dx} = 0$

$\therefore \quad 2x(x-3) + (x-3)^2 = 0 \quad \Rightarrow (x-3)(2x + x - 3) = 0$

$\Rightarrow \quad (x-3)(3x-3) = 0 \quad \Rightarrow 3(x-3)(x-1) = 0$

$\therefore \quad x = 1, 3$

$\therefore$ Possible intervals are $(-\infty, 1), (1, 3), (3, \infty)$

$$\frac{dy}{dx} = (x-3)(x-1)$$

For $(-\infty, 1) = (-)(-) = (+)$ increasing

For $(1, 3) = (-)(+) = (-)$ decreasing

For $(3, \infty) = (+)(+) = (+)$ increasing

So the function decreases in $(1, 3)$ or $1 < x < 3$

Hence, the correct option is (*a*).

Q49. The function $f(x) = 4\sin^3 x - 6\sin^2 x + 12\sin x + 100$ is strictly

(*a*) increasing in $\left(\pi, \frac{3\pi}{2}\right)$ (*b*) decreasing in $\left(\frac{\pi}{2}, \pi\right)$

(*c*) decreasing in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ (*d*) decreasing in $\left[0, \frac{\pi}{2}\right]$

Sol. Here,

$$f(x) = 4\sin^3 x - 6\sin^2 x + 12\sin x + 100$$

$$f'(x) = 12\sin^2 x \cdot \cos x - 12\sin x \cos x + 12\cos x$$

$$= 12\cos x[\sin^2 x - \sin x + 1]$$

$$= 12\cos x[\sin^2 x + (1 - \sin x)]$$

$\because \quad 1 - \sin x \geq 0$ and $\sin^2 x \geq 0$

$\therefore \quad \sin^2 x + 1 - \sin x \geq 0$ (when $\cos x > 0$)

Hence, $f'(x) > 0$, when $\cos x > 0$ i.e., $x \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

So, $f(x)$ is increasing where $x \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ and $f'(x) < 0$

when $\cos x < 0$ i.e. $x \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$

Hence, $f(x)$ is decreasing when $x \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$

As $\left(\frac{\pi}{2}, \pi\right) \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$

So $f(x)$ is decreasing in $\left(\frac{\pi}{2}, \pi\right)$

Hence, the correct option is (*b*).

Q50. Which of the following functions is decreasing in $\left(0, \frac{\pi}{2}\right)$?

(*a*) $\sin 2x$ (*b*) $\tan x$ (*c*) $\cos x$ (*d*) $\cos 3x$

Sol. Here, Let $f(x) = \cos x$; So, $f'(x) = -\sin x$

$f'(x) < 0$ in $\left(0, \frac{\pi}{2}\right)$

So $f(x) = \cos x$ is decreasing in $\left(0, \frac{\pi}{2}\right)$

Hence, the correct option is (*c*).

Q51. The function $f(x) = \tan x - x$

(*a*) always increases (*b*) always decreases

(*c*) never increases

(*d*) sometimes increases and sometimes decreases.

Sol. Here, $f(x) = \tan x - x$ So, $f'(x) = \sec^2 x - 1$

$f'(x) > 0 \ \forall \ x \in \text{R}$

So $f(x)$ is always increasing

Hence, the correct option is (*a*).

Q52. If x is real, the minimum value of $x^2 - 8x + 17$ is

(*a*) -1 (*b*) 0 (*c*) 1 (*d*) 2

Sol. Let $f(x) = x^2 - 8x + 17$

$f'(x) = 2x - 8$

For local maxima and local minima, $f'(x) = 0$

$\therefore \quad 2x - 8 = 0 \Rightarrow x = 4$

So, $x = 4$ is the point of local maxima and local minima.

$f''(x) = 2 > 0$ minima at $x = 4$

$\therefore \quad f(x)_{x=4} = (4)^2 - 8(4) + 17$

$= 16 - 32 + 17 = 33 - 32 = 1$

So the minimum value of the function is 1

Hence, the correct option is (*c*).

Q53. The smallest value of the polynomial $x^3 - 18x^2 + 96x$ in [0, 9] is:

(*a*) 126 (*b*) 0 (*c*) 135 (*d*) 160

Sol. Let $f(x) = x^3 - 18x^2 + 96x$; So, $f'(x) = 3x^2 - 36x + 96$

For local maxima and local minima $f'(x) = 0$

$\therefore \quad 3x^2 - 36x + 96 = 0$

$\Rightarrow \quad x^2 - 12x + 32 = 0 \Rightarrow x^2 - 8x - 4x + 32 = 0$

$\Rightarrow \quad x(x-8) - 4(x-8) = 0 \Rightarrow (x-8)(x-4) = 0$

$\therefore \quad x = 8, 4 \in [0, 9]$

So, $x = 4, 8$ are the points of local maxima and local minima.

Now we will calculate the absolute maxima or absolute minima at $x = 0, 4, 8, 9$

$\therefore \quad f(x) = x^3 - 18x^2 + 96x$

$f(x)_{x=0} = 0 - 0 + 0 = 0$

$f(x)_{x=4} = (4)^3 - 18(4)^2 + 96(4)$
$= 64 - 288 + 384 = 448 - 288 = 160$
$f(x)_{x=8} = (8)^3 - 18(8)^2 + 96(8)$
$= 512 - 1152 + 768 = 1280 - 1152 = 128$
$f(x)_{x=9} = (9)^3 - 18(9)^2 + 96(9)$
$= 729 - 1458 + 864 = 1593 - 1458 = 135$

So, the absolute minimum value of f is 0 at $x = 0$

Hence, the correct option is (*b*).

Q54. The function $f(x) = 2x^3 - 3x^2 - 12x + 4$, has

(*a*) two points of local maximum
(*b*) two points of local minimum
(*c*) one maxima and one minima
(*d*) no maxima or minima

Sol. We have $f(x) = 2x^3 - 3x^2 - 12x + 4$

$f'(x) = 6x^2 - 6x - 12$

For local maxima and local minima $f'(x) = 0$

$\therefore \quad 6x^2 - 6x - 12 = 0$

$\Rightarrow \quad x^2 - x - 2 = 0 \quad \Rightarrow \quad x^2 - 2x + x - 2 = 0$

$\Rightarrow \quad x(x-2) + 1(x-2) = 0 \quad \Rightarrow \quad (x+1)(x-2) = 0$

$\Rightarrow \quad x = -1, 2$ are the points of local maxima and local minima

Now $\quad f''(x) = 12x - 6$

$f''(x)_{x=-1} = 12(-1) - 6 = -12 - 6 = -18 < 0$, maxima

$f''(x)_{x=2} = 12(2) - 6 = 24 - 6 = 18 > 0$ minima

So, the function is maximum at $x = -1$ and minimum at $x = 2$

Hence, the correct option is (*c*).

Q55. The maximum value of $\sin x \cos x$ is

(*a*) $\frac{1}{4}$ (*b*) $\frac{1}{2}$ (*c*) $\sqrt{2}$ (*d*) $2\sqrt{2}$

Sol. We have $f(x) = \sin x \cos x$

$\Rightarrow \quad f(x) = \frac{1}{2} \cdot 2 \sin x \cos x = \frac{1}{2} \sin 2x$

$f'(x) = \frac{1}{2} \cdot 2 \cos 2x$

$\Rightarrow \quad f'(x) = \cos 2x$

Now for local maxima and local minima $f'(x) = 0$

$\therefore \quad \cos 2x = 0$

$2x = (2n+1)\frac{\pi}{2}, \quad n \in I$

$\Rightarrow \quad x = (2n+1)\frac{\pi}{4}$

$\therefore \qquad x = \frac{\pi}{4}, \frac{3\pi}{4} \dots$

$$f''(x) = -2 \sin 2x$$

$$f''(x)_{x=\frac{\pi}{4}} = -2 \sin 2 \cdot \frac{\pi}{4} = -2 \sin \frac{\pi}{2} = -2 < 0 \text{ maxima}$$

$$f''(x)_{x=\frac{3\pi}{4}} = -2 \sin 2 \cdot \frac{3\pi}{4} = -2 \sin \frac{3\pi}{2} = 2 > 0 \text{ minima}$$

So $f(x)$ is maximum at $x = \frac{\pi}{4}$

$\therefore$ Maximum value of $f(x) = \sin \frac{\pi}{4} \cdot \cos \frac{\pi}{4} = \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} = \frac{1}{2}$

Hence, the correct option is (*b*).

Q56. At $x = \frac{5\pi}{6}$, $f(x) = 2 \sin 3x + 3 \cos 3x$ is:

(*a*) maximum (*b*) minimum

(*c*) zero (*d*) neither maximum nor minimum.

Sol. We have $f(x) = 2 \sin 3x + 3 \cos 3x$

$$f'(x) = 2 \cos 3x \cdot 3 - 3 \sin 3x \cdot 3 = 6 \cos 3x - 9 \sin 3x$$

$$f''(x) = -6 \sin 3x \cdot 3 - 9 \cos 3x \cdot 3$$

$$= -18 \sin 3x - 27 \cos 3x$$

$$f''\left(\frac{5\pi}{6}\right) = -18 \sin 3\left(\frac{5\pi}{6}\right) - 27 \cos 3\left(\frac{5\pi}{6}\right)$$

$$= -18 \sin\left(\frac{5\pi}{2}\right) - 27 \cos\left(\frac{5\pi}{2}\right)$$

$$= -18 \sin\left(2\pi + \frac{\pi}{2}\right) - 27 \cos\left(2\pi + \frac{\pi}{2}\right)$$

$$= -18 \sin \frac{\pi}{2} - 27 \cos \frac{\pi}{2} = -18 \cdot 1 - 27 \cdot 0$$

$$= -18 < 0 \text{ maxima}$$

Maximum value of $f(x)$ at $x = \frac{5\pi}{6}$

$$f\left(\frac{5\pi}{6}\right) = 2 \sin 3\left(\frac{5\pi}{6}\right) + 3 \cos 3\left(\frac{5\pi}{6}\right) = 2 \sin \frac{5\pi}{2} + 3 \cos \frac{5\pi}{2}$$

$$= 2 \sin\left(2\pi + \frac{\pi}{2}\right) + 3 \cos\left(2\pi + \frac{\pi}{2}\right) = 2 \sin \frac{\pi}{2} + 3 \cos \frac{\pi}{2} = 2$$

Hence, the correct option is (a).

Q57. Maximum slope of the curve $y = -x^3 + 3x^2 + 9x - 27$ is:

(*a*) 0 (*b*) 12 (*c*) 16 (*d*) 32

Sol. Given that $y = -x^3 + 3x^2 + 9x - 27$

$$\frac{dy}{dx} = -3x^2 + 6x + 9$$

$\therefore$ Slope of the given curve,

$$m = -3x^2 + 6x + 9 \qquad \left(\frac{dy}{dx} = m\right)$$

$$\frac{dm}{dx} = -6x + 6$$

For local maxima and local minima, $\frac{dm}{dx} = 0$

$\therefore$ $-6x + 6 = 0 \Rightarrow x = 1$

Now $\frac{d^2m}{dx^2} = -6 < 0$ maxima

$\therefore$ Maximum value of the slope at $x = 1$ is

$$m_{x=1} = -3(1)^2 + 6(1) + 9 = -3 + 6 + 9 = 12$$

Hence, the correct option is (*b*).

Q58. $f(x) = x^x$ has a stationary point at

(*a*) $x = e$ (*b*) $x = \frac{1}{e}$ (*c*) $x = 1$ (*d*) $x = \sqrt{e}$

Sol. We have $f(x) = x^x$

Taking log of both sides, we have

$$\log f(x) = x \log x$$

Differentiating both sides w.r.t. x, we get

$$\frac{1}{f(x)} \cdot f'(x) = x \cdot \frac{1}{x} + \log x \cdot 1$$

$\Rightarrow$ $f'(x) = f(x)[1 + \log x] = x^x [1 + \log x]$

To find stationary point, $f'(x) = 0$

$\therefore$ $x^x[1 + \log x] = 0$

$x^x \neq 0$ $\therefore$ $1 + \log x = 0$

$\Rightarrow$ $\log x = -1 \Rightarrow x = e^{-1} \Rightarrow x = \frac{1}{e}$

Hence, the correct option is (*b*).

Q59. The maximum value of $\left(\frac{1}{x}\right)^x$ is:

(*a*) e (*b*) e^e (*c*) $e^{1/e}$ (*d*) $\left(\frac{1}{e}\right)^{1/e}$

Sol. Let $f(x) = \left(\frac{1}{x}\right)^x$

Taking log on both sides, we get

$$\log[f(x)] = x \log \frac{1}{x}$$

$\Rightarrow$ $\log[f(x)] = x \log x^{-1} \Rightarrow \log[f(x)] = -[x \log x]$

Differentiating both sides w.r.t. x, we get

$$\frac{1}{f(x)} \cdot f'(x) = -\left[x \cdot \frac{1}{x} + \log x \cdot 1\right] = -f(x)\,[1 + \log x]$$

$$\Rightarrow \qquad f'(x) = -\left(\frac{1}{x}\right)^x [1 + \log x]$$

For local maxima and local minima $f'(x) = 0$

$$-\left(\frac{1}{x}\right)^x [1 + \log x] = 0 \quad \Rightarrow \quad \left(\frac{1}{x}\right)^x [1 + \log x] = 0$$

$$\left(\frac{1}{x}\right)^x \neq 0$$

$$\therefore \qquad 1 + \log x = 0 \Rightarrow \log x = -1 \Rightarrow x = e^{-1}$$

So, $x = \frac{1}{e}$ is the stationary point.

Now $\quad f'(x) = -\left(\frac{1}{x}\right)^x [1 + \log x]$

$$f''(x) = -\left[\left(\frac{1}{x}\right)^x \left(\frac{1}{x}\right) + (1 + \log x) \cdot \frac{d}{dx}(x)^x\right]$$

$$f''(x) = -\left[(e)^{1/e}(e) + \left(1 + \log \frac{1}{e}\right) \frac{d}{\mathrm{dx}} \left(\frac{1}{e}\right)^{1/e}\right]$$

$$x = \frac{1}{e} = -e^{\frac{1}{e}} \; 1 < 0 \text{ maxima}$$

$\therefore$ Maximum value of the function at $x = \frac{1}{e}$ is

$$f\left(\frac{1}{e}\right) = \left(\frac{1}{1/e}\right)^{1/e} = e^{1/e}$$

Hence, the correct option is (c).

Fill in the blanks in each of the following exercises 60 to 64.

Q60. The curves $y = 4x^2 + 2x - 8$ and $y = x^3 - x + 13$ touch each other at the point ________ .

Sol. We have

$$y = 4x^2 + 2x - 8 \qquad \ldots(i)$$

and

$$y = x^3 - x + 13 \qquad \ldots(ii)$$

Differentiating eq. (i) w.r.t. x, we have

$$\frac{dy}{dx} = 8x + 2 \quad \Rightarrow \quad m_1 = 8x + 2$$

[m is the slope of curve (i)]

Differentiating eq. (*ii*) w.r.t. x, we get

$$\frac{dy}{dx} = 3x^2 - 1 \quad \Rightarrow \quad m_2 = 3x^2 - 1$$

[m_2 is the slope of curve (*ii*)]

If the two curves touch each other, then $m_1 = m_2$

$\therefore \quad 8x + 2 = 3x^2 - 1$

$\Rightarrow \quad 3x^2 - 8x - 3 = 0 \quad \Rightarrow 3x^2 - 9x + x - 3 = 0$

$\Rightarrow \quad 3x(x - 3) + 1(x - 3) = 0 \quad \Rightarrow (x - 3)(3x + 1) = 0$

$\therefore \quad x = 3, \ \frac{-1}{3}$

Putting $x = 3$ in eq. (*i*), we get

$$y = 4(3)^2 + 2(3) - 8 = 36 + 6 - 8 = 34$$

So, the required point is (3, 34)

Now for $x = -\frac{1}{3}$

$$y = 4\left(\frac{-1}{3}\right)^2 + 2\left(\frac{-1}{3}\right) - 8 = 4 \times \frac{1}{9} - \frac{2}{3} - 8$$

$$= \frac{4}{9} - \frac{2}{3} - 8 = \frac{4 - 6 - 72}{9} = \frac{-74}{9}$$

$\therefore$ Other required point is $\left(-\frac{1}{3}, \frac{-74}{9}\right)$.

Hence, the required points are (3, 34) and $\left(-\frac{1}{3}, \frac{-74}{9}\right)$.

Q61. The equation of normal to the curve $y = \tan x$ at (0, 0) is ____________ .

Sol. We have $y = \tan x$. So, $\frac{dy}{dx} = \sec^2 x$

$\therefore$ Slope of the normal $= \frac{-1}{\sec^2 x} = -\cos^2 x$

at the point (0, 0) the slope $= -\cos^2(0) = -1$

So the equation of normal at (0, 0) is $y - 0 = -1(x - 0)$

$\Rightarrow \quad y = -x \quad \Rightarrow y + x = 0$

Hence, the required equation is $y + x = 0$.

Q62. The values of a for which the function $f(x) = \sin x - ax + b$ increases on **R** are ______________ .

Sol. We have $\quad f(x) = \sin x - ax + b \quad \Rightarrow f'(x) = \cos x - a$

For increasing the function $f'(x) > 0$

$\therefore \quad \cos x - a > 0$

Since $\quad \cos x \in [-1, 1]$

$\therefore \quad a < -1 \quad \Rightarrow \quad a \in (-\infty, -1)$

Hence, the value of a is $(-\infty, -1)$.

Q63. The function $f(x) = \dfrac{2x^2 - 1}{x^4}$, $x > 0$, decreases in the interval __________ .

Sol. We have $f(x) = \dfrac{2x^2 - 1}{x^4}$

$$f'(x) = \frac{x^4(4x) - (2x^2 - 1)\cdot 4x^3}{x^8}$$

$$\Rightarrow f'(x) = \frac{4x^5 - (2x^2 - 1)\cdot 4x^3}{x^8} = \frac{4x^3[x^2 - 2x^2 + 1]}{x^8} = \frac{4(-x^2 + 1)}{x^5}$$

For decreasing the function $f'(x) < 0$

$$\therefore \quad \frac{4(-x^2 + 1)}{x^5} < 0 \quad \Rightarrow \quad -x^2 + 1 < 0 \quad \Rightarrow \quad x^2 > 1$$

$$\therefore \quad x > \pm 1 \quad \Rightarrow \quad x \in (1, \infty)$$

Hence, the required interval is $(1, \infty)$.

Q64. The least value of the function $f(x) = ax + \dfrac{b}{x}$ (where $a > 0$, $b > 0, x > 0$) is __________ .

Sol. Here, $f(x) = ax + \dfrac{b}{x} \quad \Rightarrow f'(x) = a - \dfrac{b}{x^2}$

For maximum and minimum value $f'(x) = 0$

$$\therefore \quad a - \frac{b}{x^2} = 0 \quad \Rightarrow \quad x^2 = \frac{b}{a} \quad \Rightarrow \quad x = \pm\sqrt{\frac{b}{a}}$$

Now $\quad f''(x) = \dfrac{2b}{x^3}$

$$f''(x)_{x=\sqrt{\frac{b}{a}}} = \frac{2b}{\left(\frac{b}{a}\right)^{3/2}} = 2\frac{a^{3/2}}{b^{1/2}} > 0 \qquad (\because \quad a, b > 0)$$

Hence, minima

So the least value of the function at $x = \sqrt{\dfrac{b}{a}}$ is

$$f\left(\sqrt{\frac{b}{a}}\right) = a\cdot\sqrt{\frac{b}{a}} + \frac{b}{\sqrt{\frac{b}{a}}} = \sqrt{ab} + \sqrt{ab} = 2\sqrt{ab}$$

Hence, least value is $2\sqrt{ab}$.