10. Prove that the logarithmic function is strictly increasing on $(0, \infty)$.

11. Prove that the function f given by $f(x) = x^2 - x + 1$ is neither strictly increasing nor strictly decreasing on $(-1, 1)$.

12. Which of the following functions are strictly decreasing on $\left(0, \frac{\pi}{2}\right)$?

(A) $\cos x$ (B) $\cos 2x$ (C) $\cos 3x$ (D) $\tan x$

13. On which of the following intervals is the function f given by $f(x) = x^{100} + \sin x - 1$ strictly decreasing ?

(A) $(0,1)$ (B) $\left(\frac{\pi}{2}, \pi\right)$ (C) $\left(0, \frac{\pi}{2}\right)$ (D) None of these

14. Find the least value of a such that the function f given by $f(x) = x^2 + ax + 1$ is strictly increasing on $(1, 2)$.

15. Let I be any interval disjoint from $(-1, 1)$. Prove that the function f given by $f(x) = x + \frac{1}{x}$ is strictly increasing on I.

16. Prove that the function f given by $f(x) = \log \sin x$ is strictly increasing on $\left(0, \frac{\pi}{2}\right)$ and strictly decreasing on $\left(\frac{\pi}{2}, \pi\right)$.

17. Prove that the function f given by $f(x) = \log \cos x$ is strictly decreasing on $\left(0, \frac{\pi}{2}\right)$ and strictly increasing on $\left(\frac{\pi}{2}, \pi\right)$.

18. Prove that the function given by $f(x) = x^3 - 3x^2 + 3x - 100$ is increasing in **R**.

19. The interval in which $y = x^2 e^{-x}$ is increasing is

(A) $(-\infty, \infty)$ (B) $(-2, 0)$ (C) $(2, \infty)$ (D) $(0, 2)$

6.4 Tangents and Normals

In this section, we shall use differentiation to find the equation of the tangent line and the normal line to a curve at a given point.

Recall that the equation of a straight line passing through a given point (x_0, y_0) having finite slope m is given by

$$y - y_0 = m(x - x_0)$$

Note that the slope of the tangent to the curve $y = f(x)$ at the point (x_0, y_0) is given by $\left.\frac{dy}{dx}\right]_{(x_0, y_0)} (= f'(x_0))$. So the equation of the tangent at (x_0, y_0) to the curve $y = f(x)$ is given by

$$y - y_0 = f'(x_0)(x - x_0)$$

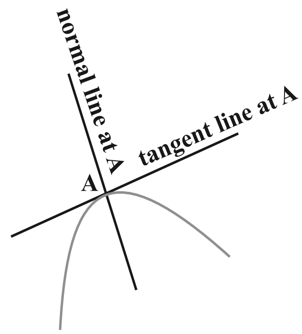

Fig 6.7

Also, since the normal is perpendicular to the tangent, the slope of the normal to the curve $y = f(x)$ at (x_0, y_0) is $\frac{-1}{f'(x_0)}$, if $f'(x_0) \neq 0$. Therefore, the equation of the normal to the curve $y = f(x)$ at (x_0, y_0) is given by

$$y - y_0 = \frac{-1}{f'(x_0)}(x - x_0)$$

i.e. $$(y - y_0) f'(x_0) + (x - x_0) = 0$$

> **Note** If a tangent line to the curve $y = f(x)$ makes an angle θ with x-axis in the positive direction, then $\frac{dy}{dx}$ = slope of the tangent = $\tan\theta$.

Particular cases

(i) If slope of the tangent line is zero, then $\tan\theta = 0$ and so $\theta = 0$ which means the tangent line is parallel to the x-axis. In this case, the equation of the tangent at the point (x_0, y_0) is given by $y = y_0$.

(ii) If $\theta \to \frac{\pi}{2}$, then $\tan\theta \to \infty$, which means the tangent line is perpendicular to the x-axis, i.e., parallel to the y-axis. In this case, the equation of the tangent at (x_0, y_0) is given by $x = x_0$ (Why?).

Example 14 Find the slope of the tangent to the curve $y = x^3 - x$ at $x = 2$.

Solution The slope of the tangent at $x = 2$ is given by

$$\left.\frac{dy}{dx}\right]_{x=2} = \left.3x^2 - 1\right]_{x=2} = 11.$$

Example 15 Find the point at which the tangent to the curve $y = \sqrt{4x-3} - 1$ has its slope $\frac{2}{3}$.

Solution Slope of tangent to the given curve at (x, y) is

$$\frac{dy}{dx} = \frac{1}{2}(4x-3)^{\frac{-1}{2}} 4 = \frac{2}{\sqrt{4x-3}}$$

The slope is given to be $\frac{2}{3}$.

So $$\frac{2}{\sqrt{4x-3}} = \frac{2}{3}$$

or $$4x - 3 = 9$$

or $$x = 3$$

Now $y = \sqrt{4x-3} - 1$. So when $x = 3$, $y = \sqrt{4(3)-3} - 1 = 2$.

Therefore, the required point is (3, 2).

Example 16 Find the equation of all lines having slope 2 and being tangent to the curve $y + \frac{2}{x-3} = 0$.

Solution Slope of the tangent to the given curve at any point (x,y) is given by

$$\frac{dy}{dx} = \frac{2}{(x-3)^2}$$

But the slope is given to be 2. Therefore

$$\frac{2}{(x-3)^2} = 2$$

or $$(x-3)^2 = 1$$

or $$x - 3 = \pm 1$$

or $$x = 2, 4$$

Now $x = 2$ gives $y = 2$ and $x = 4$ gives $y = -2$. Thus, there are two tangents to the given curve with slope 2 and passing through the points (2, 2) and (4, – 2). The equation of tangent through (2, 2) is given by

$$y - 2 = 2(x - 2)$$

or $$y - 2x + 2 = 0$$

and the equation of the tangent through (4, – 2) is given by

$$y - (-2) = 2(x - 4)$$

or $$y - 2x + 10 = 0$$

Example 17 Find points on the curve $\frac{x^2}{4}+\frac{y^2}{25}=1$ at which the tangents are (i) parallel to x-axis (ii) parallel to y-axis.

Solution Differentiating $\frac{x^2}{4}+\frac{y^2}{25}=1$ with respect to x, we get

$$\frac{x}{2}+\frac{2y}{25}\frac{dy}{dx} = 0$$

or
$$\frac{dy}{dx} = \frac{-25}{4}\frac{x}{y}$$

(i) Now, the tangent is parallel to the x-axis if the slope of the tangent is zero which gives $\frac{-25}{4}\frac{x}{y}=0$. This is possible if $x = 0$. Then $\frac{x^2}{4}+\frac{y^2}{25}=1$ for $x = 0$ gives $y^2 = 25$, i.e., $y = \pm 5$.

Thus, the points at which the tangents are parallel to the x-axis are (0, 5) and (0, – 5).

(ii) The tangent line is parallel to y-axis if the slope of the normal is 0 which gives $\frac{4y}{25x}=0$, i.e., $y = 0$. Therefore, $\frac{x^2}{4}+\frac{y^2}{25}=1$ for $y = 0$ gives $x = \pm 2$. Hence, the points at which the tangents are parallel to the y-axis are (2, 0) and (–2, 0).

Example 18 Find the equation of the tangent to the curve $y=\frac{x-7}{(x-2)(x-3)}$ at the point where it cuts the x-axis.

Solution Note that on x-axis, $y = 0$. So the equation of the curve, when $y = 0$, gives $x = 7$. Thus, the curve cuts the x-axis at (7, 0). Now differentiating the equation of the curve with respect to x, we obtain

$$\frac{dy}{dx} = \frac{1-y(2x-5)}{(x-2)(x-3)} \qquad \text{(Why?)}$$

or
$$\left.\frac{dy}{dx}\right]_{(7,0)} = \frac{1-0}{(5)(4)}=\frac{1}{20}$$

Therefore, the slope of the tangent at (7, 0) is $\frac{1}{20}$. Hence, the equation of the tangent at (7, 0) is

$$y - 0 = \frac{1}{20}(x - 7) \quad \text{or} \quad 20y - x + 7 = 0$$

Example 19 Find the equations of the tangent and normal to the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 2$ at (1, 1).

Solution Differentiating $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 2$ with respect to x, we get

$$\frac{2}{3}x^{\frac{-1}{3}} + \frac{2}{3}y^{\frac{-1}{3}}\frac{dy}{dx} = 0$$

or
$$\frac{dy}{dx} = -\left(\frac{y}{x}\right)^{\frac{1}{3}}$$

Therefore, the slope of the tangent at (1, 1) is $\left.\frac{dy}{dx}\right]_{(1,\,1)} = -1$.

So the equation of the tangent at (1, 1) is

$$y - 1 = -1\ (x - 1) \quad \text{or} \quad y + x - 2 = 0$$

Also, the slope of the normal at (1, 1) is given by

$$\frac{-1}{\text{slope of the tangent at (1,1)}} = 1$$

Therefore, the equation of the normal at (1, 1) is

$$y - 1 = 1\ (x - 1) \quad \text{or} \quad y - x = 0$$

Example 20 Find the equation of tangent to the curve given by

$$x = a \sin^3 t\,, \qquad y = b \cos^3 t \qquad \text{... (1)}$$

at a point where $t = \frac{\pi}{2}$.

Solution Differentiating (1) with respect to t, we get

$$\frac{dx}{dt} = 3a\sin^2 t\cos t \quad \text{and} \quad \frac{dy}{dt} = -3b\cos^2 t\sin t$$

or
$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-3b\cos^2 t \sin t}{3a\sin^2 t \cos t} = \frac{-b}{a}\frac{\cos t}{\sin t}$$

Therefore, slope of the tangent at $t = \frac{\pi}{2}$ is

$$\left.\frac{dy}{dx}\right]_{t=\frac{\pi}{2}} = \frac{-b\cos\frac{\pi}{2}}{a\sin\frac{\pi}{2}} = 0$$

Also, when $t = \frac{\pi}{2}$, $x = a$ and $y = 0$. Hence, the equation of tangent to the given curve at $t = \frac{\pi}{2}$, i.e., at $(a, 0)$ is

$$y - 0 = 0\,(x - a), \text{ i.e., } y = 0.$$

EXERCISE 6.3

1. Find the slope of the tangent to the curve $y = 3x^4 - 4x$ at $x = 4$.
2. Find the slope of the tangent to the curve $y = \frac{x-1}{x-2}$, $x \neq 2$ at $x = 10$.
3. Find the slope of the tangent to curve $y = x^3 - x + 1$ at the point whose x-coordinate is 2.
4. Find the slope of the tangent to the curve $y = x^3 - 3x + 2$ at the point whose x-coordinate is 3.
5. Find the slope of the normal to the curve $x = a\cos^3\theta$, $y = a\sin^3\theta$ at $\theta = \frac{\pi}{4}$.
6. Find the slope of the normal to the curve $x = 1 - a\sin\theta$, $y = b\cos^2\theta$ at $\theta = \frac{\pi}{2}$.
7. Find points at which the tangent to the curve $y = x^3 - 3x^2 - 9x + 7$ is parallel to the x-axis.
8. Find a point on the curve $y = (x - 2)^2$ at which the tangent is parallel to the chord joining the points (2, 0) and (4, 4).

9. Find the point on the curve $y = x^3 - 11x + 5$ at which the tangent is $y = x - 11$.

10. Find the equation of all lines having slope -1 that are tangents to the curve $y = \frac{1}{x-1}$, $x \neq 1$.

11. Find the equation of all lines having slope 2 which are tangents to the curve $y = \frac{1}{x-3}$, $x \neq 3$.

12. Find the equations of all lines having slope 0 which are tangent to the curve $y = \frac{1}{x^2 - 2x + 3}$.

13. Find points on the curve $\frac{x^2}{9} + \frac{y^2}{16} = 1$ at which the tangents are

(i) parallel to x-axis (ii) parallel to y-axis.

14. Find the equations of the tangent and normal to the given curves at the indicated points:

(i) $y = x^4 - 6x^3 + 13x^2 - 10x + 5$ at (0, 5)

(ii) $y = x^4 - 6x^3 + 13x^2 - 10x + 5$ at (1, 3)

(iii) $y = x^3$ at (1, 1)

(iv) $y = x^2$ at (0, 0)

(v) $x = \cos t$, $y = \sin t$ at $t = \frac{\pi}{4}$

15. Find the equation of the tangent line to the curve $y = x^2 - 2x + 7$ which is

(a) parallel to the line $2x - y + 9 = 0$

(b) perpendicular to the line $5y - 15x = 13$.

16. Show that the tangents to the curve $y = 7x^3 + 11$ at the points where $x = 2$ and $x = -2$ are parallel.

17. Find the points on the curve $y = x^3$ at which the slope of the tangent is equal to the y-coordinate of the point.

18. For the curve $y = 4x^3 - 2x^5$, find all the points at which the tangent passes through the origin.

19. Find the points on the curve $x^2 + y^2 - 2x - 3 = 0$ at which the tangents are parallel to the x-axis.

20. Find the equation of the normal at the point (am^2, am^3) for the curve $ay^2 = x^3$.

21. Find the equation of the normals to the curve $y = x^3 + 2x + 6$ which are parallel to the line $x + 14y + 4 = 0$.

22. Find the equations of the tangent and normal to the parabola $y^2 = 4ax$ at the point $(at^2, 2at)$.

23. Prove that the curves $x = y^2$ and $xy = k$ cut at right angles* if $8k^2 = 1$.

24. Find the equations of the tangent and normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point (x_0, y_0).

25. Find the equation of the tangent to the curve $y = \sqrt{3x-2}$ which is parallel to the line $4x - 2y + 5 = 0$.

Choose the correct answer in Exercises 26 and 27.

26. The slope of the normal to the curve $y = 2x^2 + 3 \sin x$ at $x = 0$ is

(A) 3 (B) $\frac{1}{3}$ (C) –3 (D) $-\frac{1}{3}$

27. The line $y = x + 1$ is a tangent to the curve $y^2 = 4x$ at the point

(A) (1, 2) (B) (2, 1) (C) (1, – 2) (D) (– 1, 2)

6.5 Approximations

In this section, we will use differentials to approximate values of certain quantities.

Let $f : D \to \mathbf{R}$, $D \subset \mathbf{R}$, be a given function and let $y = f(x)$. Let Δx denote a small increment in x. Recall that the increment in y corresponding to the increment in x, denoted by Δy, is given by $\Delta y = f(x + \Delta x) - f(x)$. We define the following

(i) The differential of x, denoted by dx, is defined by $dx = \Delta x$.

(ii) The differential of y, denoted by dy, is defined by $dy = f'(x)\ dx$ or $dy = \left(\frac{dy}{dx}\right)\Delta x$.

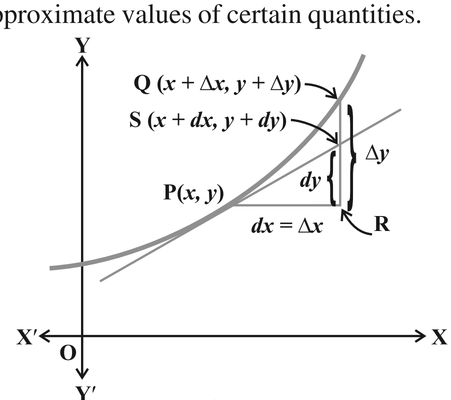

Fig 6.8

* Two curves intersect at right angle if the tangents to the curves at the point of intersection are perpendicular to each other.

In case $dx = \Delta x$ is relatively small when compared with x, dy is a good approximation of Δy and we denote it by $dy \approx \Delta y$.

For geometrical meaning of Δx, Δy, dx and dy, one may refer to Fig 6.8.

Note In view of the above discussion and Fig 6.8, we may note that the differential of the dependent variable is not equal to the increment of the variable where as the differential of independent variable is equal to the increment of the variable.

Example 21 Use differential to approximate $\sqrt{36.6}$.

Solution Take $y = \sqrt{x}$. Let $x = 36$ and let $\Delta x = 0.6$. Then

$$\Delta y = \sqrt{x + \Delta x} - \sqrt{x} = \sqrt{36.6} - \sqrt{36} = \sqrt{36.6} - 6$$

or $$\sqrt{36.6} = 6 + \Delta y$$

Now dy is approximately equal to Δy and is given by

$$dy = \left(\frac{dy}{dx}\right)\Delta x = \frac{1}{2\sqrt{x}}(0.6) = \frac{1}{2\sqrt{36}}(0.6) = 0.05 \qquad (\text{as } y = \sqrt{x})$$

Thus, the approximate value of $\sqrt{36.6}$ is $6 + 0.05 = 6.05$.

Example 22 Use differential to approximate $(25)^{\frac{1}{3}}$.

Solution Let $y = x^{\frac{1}{3}}$. Let $x = 27$ and let $\Delta x = -2$. Then

$$\Delta y = (x + \Delta x)^{\frac{1}{3}} - x^{\frac{1}{3}} = (25)^{\frac{1}{3}} - (27)^{\frac{1}{3}} = (25)^{\frac{1}{3}} - 3$$

or $$(25)^{\frac{1}{3}} = 3 + \Delta y$$

Now dy is approximately equal to Δy and is given by

$$dy = \left(\frac{dy}{dx}\right)\Delta x = \frac{1}{3x^{\frac{2}{3}}}(-2) \qquad (\text{as } y = x^{\frac{1}{3}})$$

$$= \frac{1}{3((27)^{\frac{1}{3}})^2}(-2) = \frac{-2}{27} = -0.074$$

Thus, the approximate value of $(25)^{\frac{1}{3}}$ is given by

$$3 + (-0.074) = 2.926$$

Example 23 Find the approximate value of $f(3.02)$, where $f(x) = 3x^2 + 5x + 3$.

Solution Let $x = 3$ and $\Delta x = 0.02$. Then

$$f(3.02) = f(x + \Delta x) = 3(x + \Delta x)^2 + 5(x + \Delta x) + 3$$

Note that $\Delta y = f(x + \Delta x) - f(x)$. Therefore

$$f(x + \Delta x) = f(x) + \Delta y$$

$$\approx f(x) + f'(x)\,\Delta x \qquad \text{(as } dx = \Delta x\text{)}$$

or

$$f(3.02) \approx (3x^2 + 5x + 3) + (6x + 5)\,\Delta x$$

$$= (3(3)^2 + 5(3) + 3) + (6(3) + 5)\,(0.02) \qquad \text{(as } x = 3, \Delta x = 0.02\text{)}$$

$$= (27 + 15 + 3) + (18 + 5)\,(0.02)$$

$$= 45 + 0.46 = 45.46$$

Hence, approximate value of $f(3.02)$ is 45.46.

Example 24 Find the approximate change in the volume V of a cube of side x meters caused by increasing the side by 2%.

Solution Note that

$$V = x^3$$

or

$$dV = \left(\frac{dV}{dx}\right)\Delta x = (3x^2)\,\Delta x$$

$$= (3x^2)\,(0.02x) = 0.06x^3 \text{ m}^3 \qquad \text{(as 2\% of } x \text{ is } 0.02x\text{)}$$

Thus, the approximate change in volume is $0.06\,x^3\,\text{m}^3$.

Example 25 If the radius of a sphere is measured as 9 cm with an error of 0.03 cm, then find the approximate error in calculating its volume.

Solution Let r be the radius of the sphere and Δr be the error in measuring the radius. Then $r = 9$ cm and $\Delta r = 0.03$ cm. Now, the volume V of the sphere is given by

$$V = \frac{4}{3}\pi r^3$$

or

$$\frac{dV}{dr} = 4\pi r^2$$

Therefore

$$dV = \left(\frac{dV}{dr}\right)\Delta r = (4\pi r^2)\Delta r$$

$$= 4\pi(9)^2\,(0.03) = 9.72\pi \text{ cm}^3$$

Thus, the approximate error in calculating the volume is $9.72\pi\ \text{cm}^3$.

EXERCISE 6.4

1. Using differentials, find the approximate value of each of the following up to 3 places of decimal.

(i) $\sqrt{25.3}$ (ii) $\sqrt{49.5}$ (iii) $\sqrt{0.6}$

(iv) $(0.009)^{\frac{1}{3}}$ (v) $(0.999)^{\frac{1}{10}}$ (vi) $(15)^{\frac{1}{4}}$

(vii) $(26)^{\frac{1}{3}}$ (viii) $(255)^{\frac{1}{4}}$ (ix) $(82)^{\frac{1}{4}}$

(x) $(401)^{\frac{1}{2}}$ (xi) $(0.0037)^{\frac{1}{2}}$ (xii) $(26.57)^{\frac{1}{3}}$

(xiii) $(81.5)^{\frac{1}{4}}$ (xiv) $(3.968)^{\frac{3}{2}}$ (xv) $(32.15)^{\frac{1}{5}}$

2. Find the approximate value of $f(2.01)$, where $f(x) = 4x^2 + 5x + 2$.

3. Find the approximate value of $f(5.001)$, where $f(x) = x^3 - 7x^2 + 15$.

4. Find the approximate change in the volume V of a cube of side x metres caused by increasing the side by 1%.

5. Find the approximate change in the surface area of a cube of side x metres caused by decreasing the side by 1%.

6. If the radius of a sphere is measured as 7 m with an error of 0.02 m, then find the approximate error in calculating its volume.

7. If the radius of a sphere is measured as 9 m with an error of 0.03 m, then find the approximate error in calculating its surface area.

8. If $f(x) = 3x^2 + 15x + 5$, then the approximate value of $f(3.02)$ is

(A) 47.66 (B) 57.66 (C) 67.66 (D) 77.66

9. The approximate change in the volume of a cube of side x metres caused by increasing the side by 3% is

(A) $0.06\ x^3\ m^3$ (B) $0.6\ x^3\ m^3$ (C) $0.09\ x^3\ m^3$ (D) $0.9\ x^3\ m^3$

6.6 Maxima and Minima

In this section, we will use the concept of derivatives to calculate the maximum or minimum values of various functions. In fact, we will find the 'turning points' of the graph of a function and thus find points at which the graph reaches its highest (or