Choose the correct answer in the Exercises 11 and 12.

11. If A, B are symmetric matrices of same order, then AB – BA is a

(A) Skew symmetric matrix (B) Symmetric matrix

(C) Zero matrix (D) Identity matrix

12. If $A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$, and $A + A' = I$, then the value of α is

(A) $\frac{\pi}{6}$ (B) $\frac{\pi}{3}$

(C) π (D) $\frac{3\pi}{2}$

3.7 Elementary Operation (Transformation) of a Matrix

There are six operations (transformations) on a matrix, three of which are due to rows and three due to columns, which are known as *elementary operations* or *transformations*.

(i) *The interchange of any two rows or two columns*. Symbolically the interchange of i^{th} and j^{th} rows is denoted by $R_i \leftrightarrow R_j$ and interchange of i^{th} and j^{th} column is denoted by $C_i \leftrightarrow C_j$.

For example, applying $R_1 \leftrightarrow R_2$ to $A = \begin{bmatrix} 1 & 2 & 1 \\ -1 & \sqrt{3} & 1 \\ 5 & 6 & 7 \end{bmatrix}$, we get $\begin{bmatrix} -1 & \sqrt{3} & 1 \\ 1 & 2 & 1 \\ 5 & 6 & 7 \end{bmatrix}$.

(ii) *The multiplication of the elements of any row or column by a non zero number*. Symbolically, the multiplication of each element of the i^{th} row by k, where $k \neq 0$ is denoted by $R_i \to kR_i$.

The corresponding column operation is denoted by $C_i \to kC_i$

For example, applying $C_3 \to \frac{1}{7}C_3$, to $B = \begin{bmatrix} 1 & 2 & 1 \\ -1 & \sqrt{3} & 1 \end{bmatrix}$, we get $\begin{bmatrix} 1 & 2 & \frac{1}{7} \\ -1 & \sqrt{3} & \frac{1}{7} \end{bmatrix}$

(iii) *The addition to the elements of any row or column, the corresponding elements of any other row or column multiplied by any non zero number.*

Symbolically, the addition to the elements of i^{th} row, the corresponding elements of j^{th} row multiplied by k is denoted by $R_i \to R_i + kR_j$.

The corresponding column operation is denoted by $C_i \to C_i + kC_j$.

For example, applying $R_2 \to R_2 - 2R_1$, to $C = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$, we get $\begin{bmatrix} 1 & 2 \\ 0 & -5 \end{bmatrix}$.

3.8 Invertible Matrices

Definition 6 If A is a square matrix of order m, and if there exists another square matrix B of the same order m, such that AB = BA = I, then B is called the *inverse* matrix of A and it is denoted by A^{-1}. In that case A is said to be invertible.

For example, let $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$ be two matrices.

Now $AB = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$

$$= \begin{bmatrix} 4-3 & -6+6 \\ 2-2 & -3+4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Also $BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$. Thus B is the inverse of A, in other words $B = A^{-1}$ and A is inverse of B, i.e., $A = B^{-1}$

Note

1. A rectangular matrix does not possess inverse matrix, since for products BA and AB to be defined and to be equal, it is necessary that matrices A and B should be square matrices of the same order.
2. If B is the inverse of A, then A is also the inverse of B.

Theorem 3 (Uniqueness of inverse) Inverse of a square matrix, if it exists, is unique.

Proof Let $A = [a_{ij}]$ be a square matrix of order m. If possible, let B and C be two inverses of A. We shall show that B = C.

Since B is the inverse of A

$$AB = BA = I \qquad \dots (1)$$

Since C is also the inverse of A

$$AC = CA = I \qquad \dots (2)$$

Thus $B = BI = B(AC) = (BA)C = IC = C$

Theorem 4 If A and B are invertible matrices of the same order, then $(AB)^{-1} = B^{-1} A^{-1}$.

Proof From the definition of inverse of a matrix, we have

$$(AB)\,(AB)^{-1} = 1$$

or $\quad A^{-1}\,(AB)\,(AB)^{-1} = A^{-1}I \qquad$ (Pre multiplying both sides by A^{-1})

or $\quad (A^{-1}A)\,B\,(AB)^{-1} = A^{-1} \qquad$ (Since $A^{-1}\,I = A^{-1}$)

or $\quad IB\,(AB)^{-1} = A^{-1}$

or $\quad B\,(AB)^{-1} = A^{-1}$

or $\quad B^{-1}\,B\,(AB)^{-1} = B^{-1}\,A^{-1}$

or $\quad I\,(AB)^{-1} = B^{-1}\,A^{-1}$

Hence $\quad (AB)^{-1} = B^{-1}\,A^{-1}$

3.8.1 *Inverse of a matrix by elementary operations*

Let X, A and B be matrices of, the same order such that X = AB. In order to apply a sequence of elementary row operations on the matrix equation X = AB, we will apply these row operations simultaneously on X and on the first matrix A of the product AB on RHS.

Similarly, in order to apply a sequence of elementary column operations on the matrix equation X = AB, we will apply, these operations simultaneously on X and on the second matrix B of the product AB on RHS.

In view of the above discussion, we conclude that if A is a matrix such that A^{-1} exists, then to find A^{-1} using elementary row operations, write A = IA and apply a sequence of row operation on A = IA till we get, I = BA. The matrix B will be the inverse of A. Similarly, if we wish to find A^{-1} using column operations, then, write A = AI and apply a sequence of column operations on A = AI till we get, I = AB.

Remark In case, after applying one or more elementary row (column) operations on A = IA (A = AI), if we obtain all zeros in one or more rows of the matrix A on L.H.S., then A^{-1} does not exist.

Example 23 By using elementary operations, find the inverse of the matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$.

Solution In order to use elementary row operations we may write A = IA.

or $\quad \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A$, then $\begin{bmatrix} 1 & 2 \\ 0 & -5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} A$ (applying $R_2 \to R_2 - 2R_1$)

or $$\begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix} = \begin{bmatrix}1 & 0\\ \frac{2}{5} & \frac{-1}{5}\end{bmatrix} A \quad \left(\text{applying } R_2 \rightarrow -\frac{1}{5}R_2\right)$$

or $$\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} = \begin{bmatrix}\frac{1}{5} & \frac{2}{5}\\ \frac{2}{5} & \frac{-1}{5}\end{bmatrix} A \quad (\text{applying } R_1 \rightarrow R_1 - 2R_2)$$

Thus $$A^{-1} = \begin{bmatrix}\frac{1}{5} & \frac{2}{5}\\ \frac{2}{5} & \frac{-1}{5}\end{bmatrix}$$

Alternatively, in order to use elementary column operations, we write A = AI, i.e.,

$$\begin{bmatrix}1 & 2\\ 2 & -1\end{bmatrix} = A\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

Applying $C_2 \rightarrow C_2 - 2C_1$, we get

$$\begin{bmatrix}1 & 0\\ 2 & -5\end{bmatrix} = A\begin{bmatrix}1 & -2\\ 0 & 1\end{bmatrix}$$

Now applying $C_2 \rightarrow -\frac{1}{5}C_2$, we have

$$\begin{bmatrix}1 & 0\\ 2 & 1\end{bmatrix} = A\begin{bmatrix}1 & \frac{2}{5}\\ 0 & \frac{-1}{5}\end{bmatrix}$$

Finally, applying $C_1 \rightarrow C_1 - 2C_2$, we obtain

$$\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} = A\begin{bmatrix}\frac{1}{5} & \frac{2}{5}\\ \frac{2}{5} & \frac{-1}{5}\end{bmatrix}$$

Hence $$A^{-1} = \begin{bmatrix}\frac{1}{5} & \frac{2}{5}\\ \frac{2}{5} & \frac{-1}{5}\end{bmatrix}$$

Example 24 Obtain the inverse of the following matrix using elementary operations

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}.$$

Solution Write A = I A, i.e., $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$

or $$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \text{ (applying } R_1 \leftrightarrow R_2)$$

or $$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -5 & -8 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -3 & 1 \end{bmatrix} A \text{ (applying } R_3 \to R_3 - 3R_1)$$

or $$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & -5 & -8 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -3 & 1 \end{bmatrix} A \text{ (applying } R_1 \to R_1 - 2R_2)$$

or $$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ 5 & -3 & 1 \end{bmatrix} A \text{ (applying } R_3 \to R_3 + 5R_2)$$

or $$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ \frac{5}{2} & \frac{-3}{2} & \frac{1}{2} \end{bmatrix} A \text{ (applying } R_3 \to \frac{1}{2} R_3)$$

or $$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{-1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \\ \frac{5}{2} & \frac{-3}{2} & \frac{1}{2} \end{bmatrix} A \text{ (applying } R_1 \to R_1 + R_3)$$

or $$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} = \begin{bmatrix}\frac{1}{2}&\frac{-1}{2}&\frac{1}{2}\\-4&3&-1\\\frac{5}{2}&\frac{-3}{2}&\frac{1}{2}\end{bmatrix}A \quad (\text{applying } R_2 \to R_2 - 2R_3)$$

Hence $$A^{-1} = \begin{bmatrix}\frac{1}{2}&\frac{-1}{2}&\frac{1}{2}\\-4&3&-1\\\frac{5}{2}&\frac{-3}{2}&\frac{1}{2}\end{bmatrix}$$

Alternatively, write A = AI, i.e.,

$$\begin{bmatrix}0&1&2\\1&2&3\\3&1&1\end{bmatrix} = A\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

or $$\begin{bmatrix}1&0&2\\2&1&3\\1&3&1\end{bmatrix} = A\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix} \quad (C_1 \leftrightarrow C_2)$$

or $$\begin{bmatrix}1&0&0\\2&1&-1\\1&3&-1\end{bmatrix} = A\begin{bmatrix}0&1&0\\1&0&-2\\0&0&1\end{bmatrix} \quad (C_3 \to C_3 - 2C_1)$$

or $$\begin{bmatrix}1&0&0\\2&1&0\\1&3&2\end{bmatrix} = A\begin{bmatrix}0&1&1\\1&0&-2\\0&0&1\end{bmatrix} \quad (C_3 \to C_3 + C_2)$$

or $$\begin{bmatrix}1&0&0\\2&1&0\\1&3&1\end{bmatrix} = A\begin{bmatrix}0&1&\frac{1}{2}\\1&0&-1\\0&0&\frac{1}{2}\end{bmatrix} \quad (C_3 \to \frac{1}{2}C_3)$$

or $$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -5 & 3 & 1 \end{bmatrix} = A\begin{bmatrix} -2 & 1 & \frac{1}{2} \\ 1 & 0 & -1 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \quad (C_1 \to C_1 - 2C_2)$$

or $$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix} = A\begin{bmatrix} \frac{1}{2} & 1 & \frac{1}{2} \\ -4 & 0 & -1 \\ \frac{5}{2} & 0 & \frac{1}{2} \end{bmatrix} \quad (C_1 \to C_1 + 5C_3)$$

or $$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = A\begin{bmatrix} \frac{1}{2} & \frac{-1}{2} & \frac{1}{2} \\ -4 & 3 & -1 \\ \frac{5}{2} & \frac{-3}{2} & \frac{1}{2} \end{bmatrix} \quad (C_2 \to C_2 - 3C_3)$$

Hence $$A^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{-1}{2} & \frac{1}{2} \\ -4 & 3 & -1 \\ \frac{5}{2} & \frac{-3}{2} & \frac{1}{2} \end{bmatrix}$$

Example 25 Find P^{-1}, if it exists, given $P = \begin{bmatrix} 10 & -2 \\ -5 & 1 \end{bmatrix}$.

Solution We have $P = I\,P$, i.e., $\begin{bmatrix} 10 & -2 \\ -5 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} P$.

or $$\begin{bmatrix} 1 & \frac{-1}{5} \\ -5 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{10} & 0 \\ 0 & 1 \end{bmatrix} P \quad \left(\text{applying } R_1 \to \frac{1}{10}R_1\right)$$

or $$\begin{bmatrix} 1 & \frac{-1}{5} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{10} & 0 \\ \frac{1}{2} & 1 \end{bmatrix} P \quad (\text{applying } R_2 \to R_2 + 5R_1)$$

We have all zeros in the second row of the left hand side matrix of the above equation. Therefore, P^{-1} does not exist.

EXERCISE 3.4

Using elementary transformations, find the inverse of each of the matrices, if it exists in Exercises 1 to 17.

1. $\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$ **2.** $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ **3.** $\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}$

4. $\begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}$ **5.** $\begin{bmatrix} 2 & 1 \\ 7 & 4 \end{bmatrix}$ **6.** $\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$

7. $\begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$ **8.** $\begin{bmatrix} 4 & 5 \\ 3 & 4 \end{bmatrix}$ **9.** $\begin{bmatrix} 3 & 10 \\ 2 & 7 \end{bmatrix}$

10. $\begin{bmatrix} 3 & -1 \\ -4 & 2 \end{bmatrix}$ **11.** $\begin{bmatrix} 2 & -6 \\ 1 & -2 \end{bmatrix}$ **12.** $\begin{bmatrix} 6 & -3 \\ -2 & 1 \end{bmatrix}$

13. $\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$ **14.** $\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$. **15.** $\begin{bmatrix} 2 & -3 & 3 \\ 2 & 2 & 3 \\ 3 & -2 & 2 \end{bmatrix}$

16. $\begin{bmatrix} 1 & 3 & -2 \\ -3 & 0 & -5 \\ 2 & 5 & 0 \end{bmatrix}$ **17.** $\begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$

18. Matrices A and B will be inverse of each other only if

(A) AB = BA (B) AB = BA = 0

(C) AB = 0, BA = I (D) AB = BA = I

Miscellaneous Examples

Example 26 If $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$, then prove that $A^n = \begin{bmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{bmatrix}$, $n \in \mathbf{N}$.

Solution We shall prove the result by using principle of mathematical induction.

We have $\quad P(n)$: If $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$, then $A^n = \begin{bmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{bmatrix}$, $n \in \mathbf{N}$

$$P(1) : A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}, \text{ so } A^1 = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

Therefore, the result is true for $n = 1$.

Let the result be true for $n = k$. So

$$P(k) : A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}, \text{ then } A^k = \begin{bmatrix} \cos k\theta & \sin k\theta \\ -\sin k\theta & \cos k\theta \end{bmatrix}$$

Now, we prove that the result holds for $n = k+1$

Now $\quad A^{k+1} = A \cdot A^k = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos k\theta & \sin k\theta \\ -\sin k\theta & \cos k\theta \end{bmatrix}$

$$= \begin{bmatrix} \cos\theta\cos k\theta - \sin\theta\sin k\theta & \cos\theta\sin k\theta + \sin\theta\cos k\theta \\ -\sin\theta\cos k\theta + \cos\theta\sin k\theta & -\sin\theta\sin k\theta + \cos\theta\cos k\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\theta + k\theta) & \sin(\theta + k\theta) \\ -\sin(\theta + k\theta) & \cos(\theta + k\theta) \end{bmatrix} = \begin{bmatrix} \cos(k+1)\theta & \sin(k+1)\theta \\ -\sin(k+1)\theta & \cos(k+1)\theta \end{bmatrix}$$

Therefore, the result is true for $n = k + 1$. Thus by principle of mathematical induction,

we have $A^n = \begin{bmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{bmatrix}$, holds for all natural numbers.

Example 27 If A and B are symmetric matrices of the same order, then show that AB is symmetric if and only if A and B commute, that is AB = BA.

Solution Since A and B are both symmetric matrices, therefore $A' = A$ and $B' = B$.

Let $\quad$ AB be symmetric, then $(AB)' = AB$

But $(AB)' = B'A' = BA$ (Why?)

Therefore $BA = AB$

Conversely, if AB = BA, then we shall show that AB is symmetric.

Now $(AB)' = B'A'$

$= B A$ (as A and B are symmetric)

$= AB$

Hence AB is symmetric.

Example 28 Let $A = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}, B = \begin{bmatrix} 5 & 2 \\ 7 & 4 \end{bmatrix}, C = \begin{bmatrix} 2 & 5 \\ 3 & 8 \end{bmatrix}$. Find a matrix D such that CD – AB = O.

Solution Since A, B, C are all square matrices of order 2, and CD – AB is well defined, D must be a square matrix of order 2.

Let $D = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then CD – AB = 0 gives

$$\begin{bmatrix} 2 & 5 \\ 3 & 8 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 5 & 2 \\ 7 & 4 \end{bmatrix} = O$$

or $$\begin{bmatrix} 2a+5c & 2b+5d \\ 3a+8c & 3b+8d \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 43 & 22 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

or $$\begin{bmatrix} 2a+5c-3 & 2b+5d \\ 3a+8c-43 & 3b+8d-22 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

By equality of matrices, we get

$$2a + 5c - 3 = 0 \quad \text{... (1)}$$

$$3a + 8c - 43 = 0 \quad \text{... (2)}$$

$$2b + 5d = 0 \quad \text{... (3)}$$

and $$3b + 8d - 22 = 0 \quad \text{... (4)}$$

Solving (1) and (2), we get $a = -191$, $c = 77$. Solving (3) and (4), we get $b = -110$, $d = 44$.

Therefore $$D = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -191 & -110 \\ 77 & 44 \end{bmatrix}$$

Miscellaneous Exercise on Chapter 3

1. Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, show that $(aI + bA)^n = a^n I + na^{n-1} bA$, where I is the identity matrix of order 2 and $n \in \mathbf{N}$.

2. If $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$, prove that $A^n = \begin{bmatrix} 3^{n-1} & 3^{n-1} & 3^{n-1} \\ 3^{n-1} & 3^{n-1} & 3^{n-1} \\ 3^{n-1} & 3^{n-1} & 3^{n-1} \end{bmatrix}$, $n \in \mathbf{N}$.

3. If $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$, then prove that $A^n = \begin{bmatrix} 1+2n & -4n \\ n & 1-2n \end{bmatrix}$, where n is any positive integer.

4. If A and B are symmetric matrices, prove that AB – BA is a skew symmetric matrix.

5. Show that the matrix $B'AB$ is symmetric or skew symmetric according as A is symmetric or skew symmetric.

6. Find the values of x, y, z if the matrix $A = \begin{bmatrix} 0 & 2y & z \\ x & y & -z \\ x & -y & z \end{bmatrix}$ satisfy the equation $A'A = I$.

7. For what values of x : $\begin{bmatrix} 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & 1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ x \end{bmatrix} = O$?

8. If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, show that $A^2 - 5A + 7I = 0$.

9. Find x, if $\begin{bmatrix} x & -5 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ 4 \\ 1 \end{bmatrix} = O$

10. A manufacturer produces three products x, y, z which he sells in two markets. Annual sales are indicated below:

Market	**Products**		
I	10,000	2,000	18,000
II	6,000	20,000	8,000

(a) If unit sale prices of x, y and z are ₹ 2.50, ₹ 1.50 and ₹ 1.00, respectively, find the total revenue in each market with the help of matrix algebra.

(b) If the unit costs of the above three commodities are ₹ 2.00, ₹ 1.00 and 50 paise respectively. Find the gross profit.

11. Find the matrix X so that $X\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} -7 & -8 & -9 \\ 2 & 4 & 6 \end{bmatrix}$

12. If A and B are square matrices of the same order such that AB = BA, then prove by induction that $AB^n = B^nA$. Further, prove that $(AB)^n = A^nB^n$ for all $n \in$ N.

Choose the correct answer in the following questions:

13. If $A = \begin{bmatrix} \alpha & \beta \\ \gamma & -\alpha \end{bmatrix}$ is such that $A^2 = I$, then

(A) $1 + \alpha^2 + \beta\gamma = 0$ (B) $1 - \alpha^2 + \beta\gamma = 0$

(C) $1 - \alpha^2 - \beta\gamma = 0$ (D) $1 + \alpha^2 - \beta\gamma = 0$

14. If the matrix A is both symmetric and skew symmetric, then

(A) A is a diagonal matrix (B) A is a zero matrix

(C) A is a square matrix (D) None of these

15. If A is square matrix such that $A^2 = A$, then $(I + A)^3 - 7A$ is equal to

(A) A (B) I – A (C) I (D) 3A

Summary

- A matrix is an ordered rectangular array of numbers or functions.
- A matrix having m rows and n columns is called a matrix of order $m \times n$.
- $[a_{ij}]_{m \times 1}$ is a column matrix.
- $[a_{ij}]_{1 \times n}$ is a row matrix.
- An $m \times n$ matrix is a square matrix if $m = n$.
- $A = [a_{ij}]_{m \times m}$ is a diagonal matrix if $a_{ij} = 0$, when $i \neq j$.

- $A = [a_{ij}]_{n \times n}$ is a scalar matrix if $a_{ij} = 0$, when $i \neq j$, $a_{ij} = k$, (k is some constant), when $i = j$.
- $A = [a_{ij}]_{n \times n}$ is an identity matrix, if $a_{ij} = 1$, when $i = j$, $a_{ij} = 0$, when $i \neq j$.
- A zero matrix has all its elements as zero.
- $A = [a_{ij}] = [b_{ij}] = B$ if (i) A and B are of same order, (ii) $a_{ij} = b_{ij}$ for all possible values of i and j.
- $kA = k[a_{ij}]_{m \times n} = [k(a_{ij})]_{m \times n}$
- $-A = (-1)A$
- $A - B = A + (-1) B$
- $A + B = B + A$
- $(A + B) + C = A + (B + C)$, where A, B and C are of same order.
- $k(A + B) = kA + kB$, where A and B are of same order, k is constant.
- $(k + l) A = kA + lA$, where k and l are constant.
- If $A = [a_{ij}]_{m \times n}$ and $B = [b_{jk}]_{n \times p}$, then $AB = C = [c_{ik}]_{m \times p}$, where $c_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk}$
- (i) $A(BC) = (AB)C$, (ii) $A(B + C) = AB + AC$, (iii) $(A + B)C = AC + BC$
- If $A = [a_{ij}]_{m \times n}$, then A' or $A^T = [a_{ji}]_{n \times m}$
- (i) $(A')' = A$, (ii) $(kA)' = kA'$, (iii) $(A + B)' = A' + B'$, (iv) $(AB)' = B'A'$
- A is a symmetric matrix if $A' = A$.
- A is a skew symmetric matrix if $A' = -A$.
- Any square matrix can be represented as the sum of a symmetric and a skew symmetric matrix.
- Elementary operations of a matrix are as follows:
  (i) $R_i \leftrightarrow R_j$ or $C_i \leftrightarrow C_j$
  (ii) $R_i \rightarrow kR_i$ or $C_i \rightarrow kC_i$
  (iii) $R_i \rightarrow R_i + kR_j$ or $C_i \rightarrow C_i + kC_j$
- If A and B are two square matrices such that $AB = BA = I$, then B is the inverse matrix of A and is denoted by A^{-1} and A is the inverse of B.
- Inverse of a square matrix, if it exists, is unique.