

Q. 10 If $A + B + C = 0$, then prove that
$$\begin{vmatrix} 1 & \cos C & \cos B \\ \cos C & 1 & \cos A \\ \cos B & \cos A & 1 \end{vmatrix} = 0.$$

Thinking Process

We have, given $A + B + C = 0$, so on solving the determinant by expansion, we can use $\cos(A + B) = \cos(-C)$ and similarly after simplification this expansion we will get the desired result.

Sol. We have to prove,
$$\begin{vmatrix} 1 & \cos C & \cos B \\ \cos C & 1 & \cos A \\ \cos B & \cos A & 1 \end{vmatrix} = 0$$

$$\begin{aligned} \therefore \text{LHS} &= \begin{vmatrix} 1 & \cos C & \cos B \\ \cos C & 1 & \cos A \\ \cos B & \cos A & 1 \end{vmatrix} \\ &= 1(1 - \cos^2 A) - \cos C(\cos C - \cos A \cdot \cos B) + \cos B(\cos C \cdot \cos A - \cos B) \\ &= \sin^2 A - \cos^2 C + \cos A \cdot \cos B \cdot \cos C + \cos A \cdot \cos B \cdot \cos C - \cos^2 B \\ &= \sin^2 A - \cos^2 B + 2 \cos A \cdot \cos B \cdot \cos C - \cos^2 C \\ &= -\cos(A + B) \cdot \cos(A - B) + 2 \cos A \cdot \cos B \cdot \cos C - \cos^2 C \\ &\qquad\qquad\qquad [\because \cos^2 B - \sin^2 A = \cos(A + B) \cdot \cos(A - B)] \\ &= -\cos(-C) \cdot \cos(A - B) + \cos C(2 \cos A \cdot \cos B - \cos C) \quad [\because \cos(-\theta) = \cos \theta] \\ &= -\cos C(\cos A \cdot \cos B + \sin A \cdot \sin B - 2 \cos A \cdot \cos B + \cos C) \\ &= \cos C(\cos A \cdot \cos B - \sin A \cdot \sin B - \cos C) \\ &= \cos C[\cos(A + B) - \cos C] \\ &= \cos C(\cos C - \cos C) = 0 = \text{RHS} \end{aligned}$$

Hence proved.