$$\frac{QE}{OA} = \frac{EC}{OC} \quad \text{(since } \Delta QEC \sim \Delta AOC\text{)}$$

or $$\frac{QE}{h} = \frac{r-x}{r}$$

or $$QE = \frac{h(r-x)}{r}$$

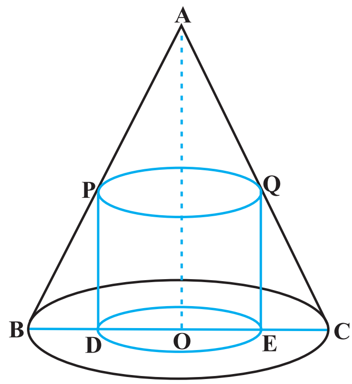

Fig 6.20

Let S be the curved surface area of the given cylinder. Then

$$S \equiv S(x) = \frac{2\pi x h(r-x)}{r} = \frac{2\pi h}{r}(rx - x^2)$$

or $$\begin{cases} S'(x) = \dfrac{2\pi h}{r}(r-2x) \\ S''(x) = \dfrac{-4\pi h}{r} \end{cases}$$

Now S′(x) = 0 gives $x = \frac{r}{2}$. Since S″(x) < 0 for all *x*, $S''\left(\frac{r}{2}\right) < 0$. So $x = \frac{r}{2}$ is a point of maxima of S. Hence, the radius of the cylinder of greatest curved surface area which can be inscribed in a given cone is half of that of the cone.

6.6.1 *Maximum and Minimum Values of a Function in a Closed Interval*

Let us consider a function *f* given by

$$f(x) = x + 2, x \in (0, 1)$$

Observe that the function is continuous on (0, 1) and neither has a maximum value nor has a minimum value. Further, we may note that the function even has neither a local maximum value nor a local minimum value.

However, if we extend the domain of *f* to the closed interval [0, 1], then *f* still may not have a local maximum (minimum) values but it certainly does have maximum value 3 = *f*(1) and minimum value 2 = *f*(0). The maximum value 3 of *f* at *x* = 1 is called *absolute maximum* value (*global maximum* or *greatest value*) of *f* on the interval [0, 1]. Similarly, the minimum value 2 of *f* at *x* = 0 is called the *absolute minimum* value (*global minimum* or *least value*) of *f* on [0, 1].

Consider the graph given in Fig 6.21 of a continuous function defined on a closed interval [*a*, *d*]. Observe that the function *f* has a local minima at *x* = *b* and local

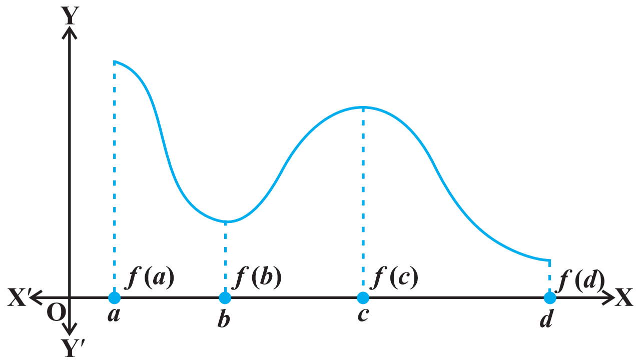

Fig 6.21

minimum value is $f(b)$. The function also has a local maxima at $x = c$ and local maximum value is $f(c)$.

Also from the graph, it is evident that f has absolute maximum value $f(a)$ and absolute minimum value $f(d)$. Further note that the absolute maximum (minimum) value of f is different from local maximum (minimum) value of f.

We will now state two results (without proof) regarding absolute maximum and absolute minimum values of a function on a closed interval I.

Theorem 5 Let f be a continuous function on an interval I = $[a, b]$. Then f has the absolute maximum value and f attains it at least once in I. Also, f has the absolute minimum value and attains it at least once in I.

Theorem 6 Let f be a differentiable function on a closed interval I and let c be any interior point of I. Then

(i) $f'(c) = 0$ if f attains its absolute maximum value at c.

(ii) $f'(c) = 0$ if f attains its absolute minimum value at c.

In view of the above results, we have the following working rule for finding absolute maximum and/or absolute minimum values of a function in a given closed interval $[a, b]$.

Working Rule

Step 1: Find all critical points of f in the interval, i.e., find points x where either $f'(x) = 0$ or f is not differentiable.

Step 2: Take the end points of the interval.

Step 3: At all these points (listed in Step 1 and 2), calculate the values of f.

Step 4: Identify the maximum and minimum values of f out of the values calculated in Step 3. This maximum value will be the absolute maximum (greatest) value of f and the minimum value will be the absolute minimum (least) value of f.

Example 39 Find the absolute maximum and minimum values of a function f given by

$$f(x) = 2x^3 - 15x^2 + 36x + 1 \text{ on the interval } [1, 5].$$

Solution We have

$$f(x) = 2x^3 - 15x^2 + 36x + 1$$

or
$$f'(x) = 6x^2 - 30x + 36 = 6(x - 3)(x - 2)$$

Note that $f'(x) = 0$ gives $x = 2$ and $x = 3$.

We shall now evaluate the value of f at these points and at the end points of the interval [1, 5], i.e., at $x = 1$, $x = 2$, $x = 3$ and at $x = 5$. So

$$f(1) = 2(1^3) - 15(1^2) + 36(1) + 1 = 24$$

$$f(2) = 2(2^3) - 15(2^2) + 36(2) + 1 = 29$$

$$f(3) = 2(3^3) - 15(3^2) + 36(3) + 1 = 28$$

$$f(5) = 2(5^3) - 15(5^2) + 36(5) + 1 = 56$$

Thus, we conclude that absolute maximum value of f on [1, 5] is 56, occurring at x =5, and absolute minimum value of f on [1, 5] is 24 which occurs at $x = 1$.

Example 40 Find absolute maximum and minimum values of a function f given by

$$f(x) = 12x^{\frac{4}{3}} - 6x^{\frac{1}{3}}, x \in [-1, 1]$$

Solution We have

$$f(x) = 12x^{\frac{4}{3}} - 6x^{\frac{1}{3}}$$

or
$$f'(x) = 16x^{\frac{1}{3}} - \frac{2}{x^{\frac{2}{3}}} = \frac{2(8x-1)}{x^{\frac{2}{3}}}$$

Thus, $f'(x) = 0$ gives $x = \frac{1}{8}$. Further note that $f'(x)$ is not defined at $x = 0$. So the critical points are $x = 0$ and $x = \frac{1}{8}$. Now evaluating the value of f at critical points $x = 0, \frac{1}{8}$ and at end points of the interval $x = -1$ and $x = 1$, we have

$$f(-1) = 12(-1)^{\frac{4}{3}} - 6(-1)^{\frac{1}{3}} = 18$$

$$f(0) = 12(0) - 6(0) = 0$$

$$f\left(\frac{1}{8}\right) = 12\left(\frac{1}{8}\right)^{\frac{4}{3}} - 6\left(\frac{1}{8}\right)^{\frac{1}{3}} = \frac{-9}{4}$$

$$f(1) = 12(1)^{\frac{4}{3}} - 6(1)^{\frac{1}{3}} = 6$$

Hence, we conclude that absolute maximum value of f is 18 that occurs at $x = -1$ and absolute minimum value of f is $\frac{-9}{4}$ that occurs at $x = \frac{1}{8}$.

Example 41 An Apache helicopter of enemy is flying along the curve given by $y = x^2 + 7$. A soldier, placed at (3, 7), wants to shoot down the helicopter when it is nearest to him. Find the nearest distance.

Solution For each value of x, the helicopter's position is at point $(x, x^2 + 7)$. Therefore, the distance between the helicopter and the soldier placed at (3,7) is

$$\sqrt{(x-3)^2 + (x^2 + 7 - 7)^2}\text{, i.e., } \sqrt{(x-3)^2 + x^4}\,.$$

Let $\qquad f(x) = (x - 3)^2 + x^4$

or $\qquad f'(x) = 2(x - 3) + 4x^3 = 2(x - 1)(2x^2 + 2x + 3)$

Thus, $f'(x) = 0$ gives $x = 1$ or $2x^2 + 2x + 3 = 0$ for which there are no real roots. Also, there are no end points of the interval to be added to the set for which f' is zero, i.e., there is only one point, namely, $x = 1$. The value of f at this point is given by $f(1) = (1 - 3)^2 + (1)^4 = 5$. Thus, the distance between the solider and the helicopter is $\sqrt{f(1)} = \sqrt{5}$.

Note that $\sqrt{5}$ is either a maximum value or a minimum value. Since

$$\sqrt{f(0)} = \sqrt{(0-3)^2 + (0)^4} = 3 > \sqrt{5},$$

it follows that $\sqrt{5}$ is the minimum value of $\sqrt{f(x)}$. Hence, $\sqrt{5}$ is the minimum distance between the soldier and the helicopter.

EXERCISE 6.5

1. Find the maximum and minimum values, if any, of the following functions given by

   (i) $f(x) = (2x - 1)^2 + 3$ (ii) $f(x) = 9x^2 + 12x + 2$

   (iii) $f(x) = -(x - 1)^2 + 10$ (iv) $g(x) = x^3 + 1$